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Large Systems of Interacting Particles and the Porous Medium Equation

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We study the empirical processes of systems of interacting particles, whose time evolution is given by coupled ordinary differential equations. In particular, as the population size tends to infinity, we derive different versions of the porous medium equation as limit dynamics. © 1990 Academic Press, Inc.

1. INTRODUCTION

In this paper we show how the porous medium equation arises in the description of certain large systems of interacting particles, whose time evolution is given by coupled ordinary differential equations. In particular, we study for each $N \in \mathbb{N}$ a set of N interacting particles in $\mathbb{R}^d$, whose positions X_N^k evolve according to

$$\frac{d}{dt} X_N^k(t) = -\frac{1}{N} \sum_{\substack{m=1 \\ m \neq k}}^{N} \nabla V_N(X_N^k(t) - X_N^m(t)), \qquad k = 1, ..., N, \tag{1.1}$$

for some sufficiently nice symmetric *interaction potential* V_N on $\mathbb{R}^d$. V_N is assumed to be obtained from some fixed function V_1 by the scaling

$$V_N(x) = \chi_N^d V_1(\chi_N x), \tag{1.2}$$

$$\chi_N = N^{\beta/d}, \qquad \beta \in (0, 1]. \tag{1.3}$$

We are interested in the bulk behaviour of the whole population of the particles, and therefore a natural object for mathematical investigation is the probability measure valued *empirical process*

$$t \to X_N(t) = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_N^k(t)},$$

where δ_a is the Dirac measure concentrated at $a \in \mathbb{R}^d$.

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Our result can be expressed in a formal way as:

> Suppose that $X_N(0)$ converges as $N \to \infty$ to the function p_0. Then $X_N(t)$ converges in the limit $N \to \infty$ for any $t \geqslant 0$ to $p(\cdot, t)$, where p is the solution of a porous medium equation with initial datum p_0.

Depending on the scaling parameter β, we obtain different versions of the porous medium equation as limit dynamics. In particular, we obtain for $\beta \in (0, 1)$ *Boussinesq's equation*

$$\frac{\partial}{\partial t} p(x, t) = G[V_1] \nabla \cdot (p(x, t) \nabla p(x, t))$$
$$= \frac{1}{2} G[V_1] \Delta p^2(x, t), \qquad p(\cdot, 0) = p_0, \tag{1.4}$$

where $G[V_1] = \int_{\mathbb{R}^d} V_1(x)\, dx$. On the other hand, we derive for $\beta = 1$ in the case $d = 1$ the generalized version

$$\frac{\partial}{\partial t} p(x, t) = \frac{1}{2} (p(x, t) F[V_1, p(x, t)])'', \qquad p(\cdot, 0) = p_0, \tag{1.5}$$

with

$$F[V_1, p] = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left(-\frac{m}{p}\right) V_1'\left(\frac{m}{p}\right). \tag{1.6}$$

Our result in this latter case describes the so-called *hydrodynamic limit* of a one-dimensional particle system with gradient dynamics. Of course, our assumptions (cf. (2.5), (3.1), (3.2)) have to assure the positivity of $G[V_1]$ and $F[V_1, p]$.

We continue with a formal derivation of our result, where we try to reveal the differences between the cases $\beta \in (0, 1)$ and $\beta = 1$. For differentiable V_1 its symmetry implies $\nabla V_1(0) = 0$, and we can write (1.1) as

$$\frac{d}{dt} X_N^k(t) = -\nabla g_N(X_N^k(t), t), \qquad k = 1, ..., N, \tag{1.7}$$

with

$$g_N(x, t) = \frac{1}{N} \sum_{m=1}^{N} V_N(x - X_N^m(t)).$$

This yields a description of X_N by a system of integral equations

$$\begin{aligned}\langle X_N(t), f(\cdot, t)\rangle &= \frac{1}{N}\sum_{k=1}^{N} f(X_N^k(t), t)\\ &= \langle X_N(0), f(\cdot, 0)\rangle\\ &\quad + \int_0^t \left\langle X_N(s), (-\nabla g_N(\cdot, s))\nabla f(\cdot, s) + \frac{\partial}{\partial s} f(\cdot, s)\right\rangle ds,\\ &\qquad f \in C_b^1(\mathbb{R}^d \times [0, \infty)), \quad t \geqslant 0. \end{aligned} \tag{1.8}$$

Here we used the abbreviation $\langle \mu, h\rangle = \int_{\mathbb{R}^d} h(x)\,\mu(dx)$ for a measure μ and a real-valued function h on $\mathbb{R}^d$.

Next, we assume the existence of a function p: $\mathbb{R}^d \times [0, \infty) \to \mathbb{R}_+$, such that

$$\lim_{N\to\infty} \langle X_N(t), f\rangle = \langle p(\cdot, t), f\rangle, \qquad f \in C_b(\mathbb{R}^d), \quad t \geqslant 0; \tag{1.9}$$

i.e., $p(x, t)$ is the *asymptotic density* of the particles near x at time t. Our aim is the determination of the dynamics of $t \to p(\cdot, t)$. We realize by (1.8) that the main obstacle is the identification of the asymptotic behaviour as $N \to \infty$ of the term $\int_0^t \langle X_N(s), \nabla g_N(\cdot, s)\nabla f(\cdot)\rangle\, ds$, in which X_N enters in a nonlinear way.

Formally we can obtain a more handsome version of this expression, where for simplicity we assume $d = 1$.

$$\begin{aligned}&\langle X_N(s), g_N'(\cdot, s) f'\rangle\\ &= \frac{1}{N^2}\sum_{k,l=1}^{N} V_N'(X_N^k(s) - X_N^l(s)) f'(X_N^k(s))\\ &= \frac{1}{2}\frac{1}{N^2}\sum_{k,l=1}^{N} V_N'(X_N^k(s) - X_N^l(s))\,[f'(X_N^k(s)) - f'(X_N^l(s))]\\ &\qquad \text{(by the symmetry of } V_N\text{)}\\ &\approx \frac{1}{2}\frac{1}{N^2}\sum_{k,l=1}^{N} V_N'(X_N^k(s) - X_N^l(s))(X_N^k(s) - X_N^l(s)) f''(X_N^k(s)). \end{aligned} \tag{1.10}$$

In our framework the particles typically are repelling each other. Therefore one may hope that they tend to be distributed fairly regularly in $\mathbb{R}$; i.e., as $N \to \infty$ the distances between neighbouring particles near x at time s may be expected to be approximately $(Np(x, s))^{-1}$. To give some meaning to this statement, we assume that the neighbours of any fixed particle at position $X_N^k(s)$ are located at

$$X_N^{k+l}(s) \approx X_N^k(s) + l(Np(X_N^k(s), s))^{-1}, \qquad l = \cdots, -1, 0, 1, \cdots. \tag{1.11}$$

Inserting (1.2), (1.3) and this assumption, which usually is called the *hypothesis of local equilibrium* (cf. Section 4.B) into (1.10) yields

$$\begin{aligned}&\langle X_N(s), g'_N(\cdot, s)f'\rangle\\&\quad\approx\frac{1}{2}\frac{1}{N}\sum_{k=1}^{N} f''(X_N^k(s))\frac{N^\beta}{N}\sum_{l=-\infty}^{\infty} V_1'\left(\frac{N^\beta l}{Np(X_N^k(s), s)}\right)\frac{N^\beta l}{Np(X_N^k(s), s)}\\&\quad=\frac{1}{2}\left\langle X_N(s), \left(\frac{N^\beta}{N}\sum_{l=-\infty}^{\infty} V_1'\left(\frac{N^\beta l}{Np(\cdot, s)}\right)\frac{N^\beta l}{Np(\cdot, s)}\right)f''\right\rangle.\end{aligned}$$

We have extended the l-summation from $-\infty$ to ∞, since this contributes only a negligible error, if V_1 is decreasing sufficiently fast at infinity.

At this point the differences between the cases $\beta\in(0, 1)$ and $\beta=1$ appear. If $\beta\in(0, 1)$, then

$$\begin{aligned}&\frac{N^\beta}{N}\sum_{l=-\infty}^{\infty} V_1'\left(\frac{N^\beta l}{Np(\cdot, s)}\right)\frac{N^\beta l}{Np(\cdot, s)}\\&\quad\approx\int_{\mathbb{R}} V_1'\left(\frac{x}{p(\cdot, s)}\right)\frac{x}{p(\cdot, s)}\,dx\\&\quad=p(\cdot, s)\int_{\mathbb{R}} xV_1'(x)\,dx=-p(\cdot, s)\,G[V_1],\end{aligned}$$

whereas we obtain for $\beta=1$

$$\frac{N^\beta}{N}\sum_{l=-\infty}^{\infty} V_1'\left(\frac{N^\beta l}{Np(\cdot, s)}\right)\frac{N^\beta l}{Np(\cdot, s)}=-F[V_1, p(\cdot, s)]. \tag{1.12}$$

Therefore, (1.8)–(1.10) yield the limit equations

$$\begin{aligned}\langle p(\cdot, t), f\rangle=\langle p_0, f\rangle+\frac{1}{2}G[V_1]\int_0^t\langle p(\cdot, s)^2, f''\rangle\,ds,&\\ f\in C_b^2(\mathbb{R}),\quad t\geqslant 0,&\end{aligned} \tag{1.13}$$

for $\beta\in(0, 1)$, resp.

$$\begin{aligned}\langle p(\cdot, t), f\rangle=\langle p_0, f\rangle+\frac{1}{2}\int_0^t\langle p(\cdot, s), F[V_1, p(\cdot, s)]f''\rangle\,ds,&\\ f\in C_b^2(\mathbb{R}),\quad t\geqslant 0,&\end{aligned} \tag{1.14}$$

if $\beta=1$. These equations are weak versions of (1.4), (1.5).

This paper is organized as follows. In Section 2, resp. 3, we present the exact formulations of our results in the cases $\beta\in(0, 1)$, resp. $\beta=1$. Section 4 contains a discussion of these results. Finally, the proofs can be found in Section 5, resp. 6.

2. The Case $\beta \in (0, 1)$: Convergence to Boussinesq's Equation

First we collect the assumptions on the interaction potential V_1. Then we discuss some analytic aspects of Boussinesq's equation, which are relevant in our context. In a third subsection we present our result.

A. *Assumptions on* V_1

To assure unique existence of the solutions X_N^k, $k = 1, \ldots, N$, of the system (1.1), we first assume

$$V_1 \in C_b^2(\mathbb{R}^d). \tag{2.1}$$

Furthermore, we assume that V_1 can be written as a convolution product

$$V_1(x) = \int_{\mathbb{R}^d} W_1(y)\, W_1^-(x-y)\, dy = (W_1 * W_1^-)(x) \tag{2.2}$$

of a real-valued function W_1 on $\mathbb{R}^d$ with its reflected version $W_1^-(x) = W_1(-x)$.

Let

$$U_{1,k_1,\ldots,k_d;j}^{[d]}(x) = (-1)^{1+k_1+\cdots+k_d} \frac{x_1^{k_1}\cdots x_d^{k_d}}{k_1!\cdots k_d!} (\partial_j W_1)(x),$$

$$k_1, \ldots, k_d = 0, \ldots, \left[\frac{d+2}{2}\right] + 1,$$

$$k_1 + \cdots + k_d \leqslant \left[\frac{d+2}{2}\right] + 1, \qquad j = 1, \ldots, d.$$

We assume that the functions W_1 and $U_{1,k_1,\ldots,k_d;j}^{[d]}$ satisfy

$$\nabla \widetilde{W}_1 \in L^\infty(\mathbb{R}^d), \tag{2.3}$$

$$|W_1(x)| \leqslant C(1+|x|)^{-d-2}, \tag{2.4}$$

$$0 < \int_{\mathbb{R}^d} W_1(x)\, dx < \infty, \tag{2.5}$$

$$|\widetilde{U_{1,k_1,\ldots,k_d;j}^{[d]}}(\mu)| \leqslant C\, |\widetilde{W}_1(\mu)|,$$

$$k_1, \ldots, k_d = 0, \ldots, \left[\frac{d+2}{2}\right],$$

$$1 \leqslant k_1 + \cdots + k_d \leqslant \left[\frac{d+2}{2}\right], \qquad j = 1, \ldots, d, \tag{2.6}$$

$$|U^{[d]}_{1,k_1,\ldots,k_d;j}(x)| \leqslant C(1+|x|)^{-(d+1)/2},$$
$$k_1, \ldots, k_d = 0, \ldots, \left[\frac{d+2}{2}\right]+1,$$
$$k_1 + \cdots + k_d = \left[\frac{d+2}{2}\right]+1, \qquad j=1, \ldots, d, \tag{2.7}$$

where $\tilde{f}(\mu) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\mu x} f(x)\, dx$ is the Fourier transform of some function f, and $[x] = \sup\{n \in \mathbb{Z} : n \leqslant x\}$.

Here and in the rest of this paper we use the symbol C to denote any positive constant.

When we study convergence to weak solutions of Boussinesq's equation, which are not classical solutions, we have to restrict to one dimension. In this case we need the convexity of V_1 on either side of the origin:

$$V_1 \in C_b^2(\mathbb{R}\backslash\{0\}), \qquad V_1''(x) \geqslant 0 \qquad \text{for} \quad x \neq 0. \tag{2.8}$$

Although (2.1) cannot hold now, a moment's thought shows that unique existence of the solution of (1.1) is established again, if we additionally assume

$$X_N^k(0) \neq X_N^m(0) \qquad \text{for} \quad k, m = 1, \ldots, N, \quad k \neq m. \tag{2.9}$$

Note that (2.2), (2.4), (2.8) imply

$$V_1(x), \operatorname{sgn}(-x)\, V_1'(x) \geqslant 0, \qquad x \in \mathbb{R}\backslash\{0\}. \tag{2.10}$$

In particular, the interaction is purely repulsive and has decreasing strength as the distance increases. Therefore, for any $N \in \mathbb{N}$ and $T > 0$

$$d_N = \inf\{|X_N^k(t) - X_N^m(t)| : k, m = 1, \ldots, N, k \neq m, t \leqslant T\} > 0; \tag{2.11}$$

i.e., no pair of particles will meet in the course of time, and the singularity of V_1' in 0 will not show up.

Remark. (a) Let $d=1$ and $W_1(y) = (1/\Gamma(a))\alpha^a y^{a-1} e^{-\alpha y} \mathbb{1}_{[0,\infty)}(y)$ be a gamma density. Then, assumptions (2.1), (2.3)–(2.7) are satisfied for $\alpha > 0$, $a \geqslant 2$. On the other hand, (2.3)–(2.8) are valid for $a = 1$. In this case $V_1(x) = (\alpha/2)\exp(-\alpha|x|)$ is a bilateral exponential density. By using the identity $x\delta(x) \equiv 0$ ($\delta(\cdot)$ is the *delta function*), we easily check $U^{[1]}_{1,1;1}(x) = -\alpha^2 x \exp(-\alpha x)\, \mathbb{1}_{[0,\infty)}(x)$. Hence, the functions $U^{[d]}_{1,k_1,\ldots,k_d;j}$ may be well defined, even if W_1 has a singularity at the origin.

(b) Let $d \geqslant 1$ be arbitrary and $W_1^{[r]}(x) = (-\Delta+1)^{-r}(x)$, $r = 1, 2, \ldots$, be *Green's function* of $(-\Delta+1)^r$. $W_1^{[r]}$ is positive, integrable and together with its partial derivatives exponentially decreasing at infinity. Moreover, we have $\widetilde{W_1^{[r]}}(\mu) = (2\pi)^{-d/2}(1+\mu^2)^{-r}$. Therefore, $W_1(\cdot) = W_1^{[r]}(\cdot)$ is a probability density on $\mathbb{R}^d$, and (2.1), (2.3)–(2.7) hold for $r > \max\{(d+2)/4, d/2\}$.

B. *Some Aspects of Boussinesq's Equation*

The porous medium equation, in particular Boussinesq's equation, is a prominent example of a nonlinear diffusion equation, which has been studied extensively in the literature. A survey of its characteristic features and a list of references can be found, e.g., in [3]. For our purposes it is important that it is parabolic; i.e., the diffusion coefficient is positive, in those regions of space, where its solution p is positive, and gets degenerate, whenever p tends to 0. This entails that p is smooth, whenever it is positive. On the other hand, if $p(\cdot, 0)$ has compact support, then $p(\cdot, t)$ has compact support too for any $t \geqslant 0$. In general higher derivatives of $p(\cdot, t)$ will be singular at the boundary ∂Q_t of $Q_t = \operatorname{supp}(p(\cdot, t))$. It is possible that the derivatives of order $\geqslant 2$ explode at some $t^* > 0$, even if $p(\cdot, 0)$ was smooth (cf. [1]). For $d > 1$ also $\nabla p(\cdot, t)$ can get unbounded (cf. [3]). In such situations p does not solve the classical equation (1.4) for all positive times, but rather some weak version, e.g., (1.13) or

$$\begin{aligned}
&\langle p(\cdot, t), f(\cdot, t)\rangle \\
&\quad = \langle p_0, f(\cdot, 0)\rangle + G[V_1] \int_0^t \left\langle p(\cdot, s), (-\nabla p(\cdot, s))\, \nabla f(\cdot, s) + \frac{\partial}{\partial s} f(\cdot, s)\right\rangle ds \\
&\qquad f \in C_b^1(\mathbb{R}^d \times [0, \infty)), \quad t \geqslant 0.
\end{aligned} \tag{2.12}$$

Basic properties of the solution of this equation are summarized in:

PROPOSITION (2.13). *Assume*

$$\begin{gathered}
p_0 \in C_b^\alpha(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \qquad \text{for some} \quad \alpha > 0, \\
p_0(\cdot) \geqslant 0, \quad \nabla p_0 \in L^2(\mathbb{R}^d).
\end{gathered} \tag{2.14}$$

(i) *Then* (2.12) *has a unique positive solution p satisfying*

$$\int_0^t \|\nabla p^2(\cdot, s)\|_2^2\, ds < \infty, \qquad 0 < t < \infty, \tag{2.15}$$

$$\|p(\cdot, t)\|_q \leqslant \|p_0\|_q, \qquad q \in [1, \infty], \quad 0 < t < \infty, \tag{2.16}$$

with equality for $q = 1$, and

$$p \in C_b(\mathbb{R}^d \times [0, \infty)). \tag{2.17}$$

(ii) *p is infinitely differentiable in $U = \{(x, t) \in \mathbb{R}^d \times (0, \infty) : p(x, t) > 0\}$. In particular, p satisfies the classical equation* (1.4) *in U.*

(iii) *For $d=1$, suppose additionally*

$$p_0' \in L^{\infty}(\mathbb{R}). \tag{2.18}$$

Then,

$$\|p'(\cdot, t)\|_q \leqslant \|p_0'\|_q, \qquad q \in [2, \infty], \quad 0 < t < \infty. \tag{2.19}$$

In (2.14) we denote by $C_b^{\alpha}(\mathbb{R}^d)$ the set of bounded Hőlder continuous functions on $\mathbb{R}^d$ with Hőlder exponent α.

Proof. First, we note that (2.2), (2.5) imply $0 < G[V_1] < \infty$. The unique existence of a solution p of (2.12) with

$$\int_0^t \int_{\mathbb{R}^d} (p^2(x, s) + |\nabla p^2(x, s)|^2)\, dx\, ds < \infty, \qquad 0 < t < \infty,$$

is proved in [21]. Relation (2.17) is derived in [5].

Next, (2.16), (2.19) can be shown by first proving it for the solution p_ε of the uniformly parabolic equation

$$\frac{\partial}{\partial t} p_\varepsilon(x, t) = G[V_1]\, \nabla \cdot (p_\varepsilon(x, t)\, \nabla p_\varepsilon(x, t)) + \varepsilon \Delta p_\varepsilon(x, t),$$

$$p_\varepsilon(x, 0) = (p_0 * \rho_\varepsilon)(x),$$

where $\rho_\varepsilon(x) = (2\pi\varepsilon)^{-d/2} \exp(-x^2/2\varepsilon)$, and then taking the limit $\varepsilon \to 0$.

Proposition (2.13)(ii) follows from (2.17) and the classical theory of quasilinear parabolic equations, cf. [9].

In this paper p_0, and hence $p(\cdot, t)$ too, will always be a probability density.

Unfortunately the regularity of the solution of (2.12), which is provided by Proposition (2.13), is not enough for our purposes. We shall formulate essentially two convergence results involving classical, resp. weak, solutions of Boussinesq's equation. Instead of giving additional conditions on the initial datum p_0 and using known results from the literature to derive properties of the solution p, which are advantageous to us in those different situations, we prefer to give conditions for p directly.

In our first special case, when we study the convergence to classical solutions of Boussinesq's equation, we need

$$\sup_{t \leqslant T} \sup_{\substack{0 \leqslant k_1, \ldots, k_d \leqslant [(d+2)/2]+2 \\ 1 \leqslant k_1 + \cdots + k_d \leqslant [(d+2)/2]+2}} \left\| \frac{\partial^{k_1 + \cdots + k_d}}{(\partial x_1)^{k_1} \cdots (\partial x_d)^{k_d}} p(\cdot, t) \right\|_\infty < \infty$$

$$\text{for some} \quad T \in (0, \infty). \tag{P1}$$

This condition holds, e.g., if p_0 is strictly positive and $p_0 \in C_b^{[(d+2)/2]+2}(\mathbb{R}^d)$. Condition (P1) allows an improvement of (2.15). For a classical solution p of (1.4) we obtain

$$\begin{aligned}\|\nabla p(\cdot, t)\|_2^2 &= \|\nabla p_0\|_2^2 - G[V_1] \int_0^t \langle \Delta p(\cdot, s), \Delta p^2(\cdot, s)\rangle \, ds \\ &= \|\nabla p_0\|_2^2 - 2G[V_1] \int_0^t (\langle p(\cdot, s), |\Delta p(\cdot, s)|^2\rangle \\ &\quad + \langle \Delta p(\cdot, s), |\nabla p(\cdot, s)|^2\rangle) \, ds \\ &\leqslant \|\nabla p_0\|_2^2 + C \int_0^t (1 + \|\nabla p(\cdot, s)\|_2^2) \, ds.\end{aligned}$$

Therefore, by (2.14) and Gronwall's Lemma

$$\sup_{t \leqslant T} \|\nabla p(\cdot, t)\|_2^2 < \infty. \tag{2.20}$$

Later when investigating the convergence to weak solutions of Boussinesq's equation with compact support, we have to restrict to one dimension. Then we assume in addition to (2.14)

$$\begin{aligned}&Q_t = \operatorname{supp}(p(\cdot, t)) \text{ is a bounded interval for } t \in [0, T], \\ &\sup\{|p''(x, t)| : x \in \operatorname{int}(Q_t), t \in [0, T]\} < \infty, \\ &\sup\{|p'''(x, t)| : x \in \operatorname{int}(Q_t), \operatorname{dist}(x, \partial Q_t) \geqslant \varepsilon, t \in [0, T]\} < \infty \\ &\qquad \text{for any} \quad \varepsilon > 0.\end{aligned} \tag{P2}$$

Condition (P2) is satisfied, e.g., if Q_0 is a bounded interval, and if p_0 is in C_b^3 within Q_0, and if additionally p_0 is concave in Q_0 (cf. [2, 7]), or more generally, if the boundaries of Q_t are moving at time $t = 0$ (cf. [4]).

C. *The Result for* $\beta \in (0, 1)$

To study the asymptotics of the empirical processes X_N we introduce *smooth* versions

$$\begin{aligned}h_N(x, t) &= (X_N(t) * W_N^{\#})(x), \\ W_N^{\#}(x) &= G[V_1]^{-1/2} \chi_N^d W_1(\chi_N x).\end{aligned}$$

Since (2.2) implies $G[V_1] = (\int_{\mathbb{R}^d} W_1(y) \, dy)^2$, the function $W_1^{\#}$ is a probability density, and therefore $h_N(\cdot, t)$ is also a probability density, which is obtained from $X_N(t)$ by distributing the "masses" of the particles slightly around their positions.

Additionally we define the metric

$$\|\mu-\tau\|_* = \sup\{\langle \mu-\tau, f\rangle : f \in C_b^1(\mathbb{R}^d),\ \|f\|_\infty + \|\nabla f\|_\infty \leqslant 1\}$$

on the space $\mathscr{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$. This metric generates the *weak-∗-topology* on $\mathscr{P}(\mathbb{R}^d)$, i.e., the weakest topology, such that the functions $\mu \to \langle \mu, f\rangle$, $f \in C_b(\mathbb{R}^d)$, are continuous, cf. [6].

Part (i) of the following theorem describes the asymptotics of the functions h_N, whereas the second part, which gives the limit behaviour of the processes X_N directly, is a less difficult consequence of (i).

THEOREM (2.21). *Let* $\beta \in (0, 1)$. *Assume* (1.1)–(1.3), (2.2)–(2.7), (2.14),

$$\lim_{N\to\infty} \|h_N(\cdot, 0) - p_0\|_2 = 0, \tag{2.22}$$

and additionally (2.1), (P1) *for arbitrary* $d \geqslant 1$, *or* (2.8), (2.9), (2.18), (P2) *for* $d = 1$. *Then*

(i)
$$\lim_{N\to\infty} \left\{ \sup_{t\leqslant T} \|h_N(\cdot, t) - p(\cdot, t)\|_2^2 + \int_0^T \frac{1}{N} \sum_{k=1}^{N} \left| \frac{1}{N} \sum_{\substack{m=1\\ m\neq k}}^{N} \nabla V_N(X_N^k(s) - X_N^m(s)) - G[V_1]\,\nabla p(X_N^k(s), s) \right|^2 ds \right\} = 0, \tag{2.23}$$

where p is the unique solution of (2.12).

(ii)
$$\lim_{N\to\infty} \sup_{t\leqslant T} \|X_N(t) - p(\cdot, t)\|_* = 0. \tag{2.24}$$

Remark (2.25). (a) If (2.1) holds, i.e., if we can use the representation (1.7) for the dynamics (1.1), we can write (2.23) as

$$\lim_{N\to\infty} \left\{ \sup_{t\leqslant T} \|h_N(\cdot, t) - p(\cdot, t)\|_2^2 + \int_0^T \langle X_N(s), |\nabla g_N(\cdot, s) - G[V_1]\,\nabla p(\cdot, s)|^2\rangle\, ds \right\} = 0. \tag{2.26}$$

(b) In the case $d = 1$, where (2.8), (2.9) and (P2) are assumed, $p'(\cdot, s)$ is in general not well defined at the boundary ∂Q_s of the support of $p(\cdot, s)$. However, by (5.36), which also holds for X_N, g_N, h_N instead of $X_{N,a}$, $g_{N,a}$, $h_{N,a}$, and since at these two boundary points at most two particles can be located (cf. (2.11)), the asymptotics of the second term on the left side of (2.23), (2.26) or similar expressions in the rest of this paper are not influenced by a particular choice of $p'(\cdot, s)$ at ∂Q_s.

(c) In (2.23), (2.26) the function W_N is used in two different ways: First in the definition of the dynamics ($V_N = W_N * W_N^-$), and second as a *mollifier* ($h_N(\cdot, t) = G[V_1]^{-1/2} (X_N(t) * W_N)(\cdot)$).

(d) Equation (2.23) is not only a convergence result for the empirical processes X_N of the locations of the particles, but also for the empirical processes of the *velocities* $t \to X_N^v(t) = (1/N) \sum_{k=1}^N [-(1/N) \sum_{m=1, m \neq k}^N \nabla V_N(X_N^k(t) - X_N^m(t))] \; \delta_{X_N^k(t)}$. In particular, we obtain easily from (2.23)

$$\lim_{N \to \infty} \int_0^T \langle X_N^v(t), f(\cdot, t)\rangle \, dt = G[V_1] \int_0^T \langle p(\cdot, t), (-\nabla p(\cdot, t)) f(\cdot, t)\rangle \, dt,$$

$$f \in C_b(\mathbb{R}^d \times [0, T]; \mathbb{R}^d).$$

3. The Case $\beta = 1$: The Hydrodynamic Limit

We assume throughout this section $\beta = d = 1$. The present case $\beta = 1$ is distinct from the preceding case $\beta \in (0, 1)$ in at least two aspects: The heuristic interpretation of the dynamics (1.1) (cf. Section 4.A), and the limit dynamics (cf. (1.4), (1.5)). Therefore it is not surprising at all that we need fairly different assumptions on V_1 and the initial data p_0, $X_N(0)$, $N \in \mathbb{N}$.

We assume

$$V_1 \in C^4(\mathbb{R}\backslash\{0\}); \qquad V_1^{(4)}(x) > 0, \quad x \in \mathbb{R}\backslash\{0\}, \tag{3.1}$$

$$|V_1^{(k)}(x)| \leqslant C(1 + |x|)^{-2-k}, \qquad k = 0, 1, 2, \quad |x| \geqslant 1. \tag{3.2}$$

Assumptions (3.1), (3.2) are satisfied, e.g., for a bilateral exponential density $V_1(x) = (\alpha/2) \exp(-\alpha\,|x|)$, $\alpha > 0$.

Moreover, we assume

$$X_N(0) \text{ is symmetric with respect to } 0,\ X_N^1(0) \leqslant X_N^2(0) \leqslant \cdots \leqslant X_N^N(0), \tag{3.3}$$

$$X_N^k(0) - X_N^{k-1}(0) \text{ is decreasing in } k \in \left\{2, ..., \left[\frac{N+2}{2}\right]\right\}, \tag{3.4}$$

$$\inf\{N(X_N^k(0) - X_N^{k-1}(0)) : k = 2, ..., N;\ N \in \mathbb{N}\} > 0, \tag{3.5}$$

$$\sup\{N(X_N^k(0) - X_N^{k-1}(0)) : k = 2, ..., N;\ N \in \mathbb{N}\} < \infty, \tag{3.6}$$

$$\lim_{N \to \infty} \|X_N(0) - p_0\|_* = 0. \tag{3.7}$$

We can state now our result.

THEOREM (3.8). *Let $d=\beta=1$. Assume* (1.1)–(1.3), (3.1)–(3.7). *Then for any $T>0$*

$$\lim_{N\to\infty} \sup_{t\leqslant T} \|X_N(t)-p(\cdot,t)\|_* = 0, \tag{3.9}$$

where p is the unique solution of (1.14) *in* $L^1(\mathbb{R}\times[0,T])\cap L^\infty(\mathbb{R}\times[0,T])$.

4. Discussion of the Results and the Relation to Other Work

This section contains some remarks on the dynamics (1.1), on the differences between the cases $\beta\in(0,1)$ and $\beta=1$, and moreover some comments on the relation to other work.

A. *Some Remarks on the Dynamics* (1.1), *Resp.* (1.7)

In the situations considered in this paper typically any particle is repelled by those other particles in the population, which are located in a certain neighbourhood. In case $\beta\in(0,1)$ this is implied by the fact that V_1 is mostly positive (since by (2.2), (2.5), $G[V_1]=\int_{\mathbb{R}^d} V_1(x)\,dx>0$), and the minus sign on the right side of (1.1). If (2.8) holds, or in case $\beta=1$ by (3.1), the interaction is even purely repulsive. The scaling (1.2), (1.3) determines both the strength and the range of this interaction, and therefore the number of different neighbours interacting with some fixed particle.

For $\beta\in(0,1)$ the function $g_N(x,t)$ formally represents the *density* of the population of the $X_N^k(t)$, $k=1,\dots,N$, near $x\in\mathbb{R}^d$ at time t, at least if $V_1\geqslant 0$ and $\int_{\mathbb{R}^d} V_1(x)\,dx=1$. This can be seen when taking V_1 as the indicator function of a ball with volume 1 and centre 0, although such a choice does not satisfy our assumptions on V_1. In this case

$$\begin{aligned} g_N(x,t) &= (1/N)\,\chi_N^d \text{ times the number of particles at time } t \text{ in the ball } B(\chi_N^{-d},x) \text{ with centre } x \text{ and volume } \chi_N^{-d} \\ &= \frac{\text{number of particles in } B(\chi_N^{-d},x)}{\text{volume of } B(\chi_N^{-d},x)}\cdot\frac{1}{\text{total number of particles}}. \end{aligned}$$

The space element $\Delta x=B(\chi_N^{-d},x)$ is *macroscopically small*, since its volume is $\chi_N^{-d}=N^{-\beta}$, which tends to 0 as $N\to\infty$. On the other hand, if the $X_N^k(t)$ are distributed "sufficiently smooth," one expects that the number of particles in $B(\chi_N^{-d},x)$ is of order $N\ \mathrm{vol}(B(\chi_N^{-d},x))=N^{1-\beta}$, which tends to ∞ as $N\to\infty$; i.e., Δx is *microscopically large.* Hence, g_N meets for $\beta\in(0,1)$ the usual heuristic picture of a *population density* or *one-particle-distribution function*, as it is called in the context of statistical physics. For a discussion of this concept cf. [19, p. 75]. In both extreme cases $\beta=0$ and $\beta=1$ the function g_N does not fit this picture. If $\beta=0$, then

$\Delta x = B(1, x)$ does not become small in the limit $N \to \infty$. For $\beta = 1$ the space element $\Delta x = B(1/N, x)$ is not microscopically large, since then the typical number of particles in Δx remains finite as $N \to \infty$.

The case $\beta = 1$ allows the following interpretation. Consider for any $N \in \mathbb{N}$ a system of N interacting particles, whose positions Y_N^k evolve according to

$$\frac{d}{dt} Y_N^k(t) = -\sum_{\substack{l=1 \\ l \neq k}}^{N} \nabla V_1(Y_N^k(t) - Y_N^l(t)), \qquad k = 1, ..., N, \tag{4.1}$$

cf. [10], where an infinite system of this type is investigated. To study the bulk properties of this system we introduce the space-time-scaling $t \to t/N^{2/d}$, $x \to x/N^{1/d}$; i.e., we consider the processes $t \to X_N^k(t) = N^{-1/d} Y_N^k(N^{2/d} t)$. By (4.1) these processes satisfy (1.1) with $\beta = 1$.

Hence, in case $\beta = 1$ the dynamics (1.1) is obtained from some fixed dynamics by a pure scaling, whereas in cases $\beta < 1$ we have a family of different time evolutions indexed by N.

The scaling describing the transition from the processes Y_N^k to the processes X_N^k, together with the subsequent limit $N \to \infty$, is called *hydrodynamic limit*, cf. [20], where the same model with additional Brownian motion in the dynamics of the Y_N^k is studied.

The case $\beta = 0$ describes so-called *weakly interacting deterministic processes*, and can be handled by quite the same methods that have been developed in the context of weakly interacting stochastic processes. In particular, by [14] the limit behaviour is given by

$$\lim_{N \to \infty} \langle X_N(t), f \rangle = \langle p_w(\cdot, t), f \rangle, \qquad t \geq 0, \quad f \in C_b(\mathbb{R}^d),$$

where p_w is the unique solution of the nonlinear integro-differential equation

$$\begin{aligned} &\langle p_w(\cdot, t), f \rangle \\ &\quad = \langle p_0, f \rangle - \int_0^t \left\langle p_w(\cdot, s), \left(\int_{\mathbb{R}^d} \nabla V_1(\cdot - y)\, p_w(y, s)\, dy \right) \nabla f \right\rangle ds, \\ &\qquad\qquad t \geq 0, \quad f \in C_b^1(\mathbb{R}^d). \end{aligned}$$

In the physical literature the limit $N \to \infty$ in the case $\beta = 0$ is called *mean-field* or *Vlasov-limit*.

Finally, note that the case $\beta > 1$ is trivial. As the *range of the interaction* decreases like $N^{-\beta/d}$, whereas typical inter-particle distances are of order $N^{-1/d}$, the interaction between different particles now vanishes in the limit $N \to \infty$, and we obtain $X_N(t) \approx X_N(0) \approx p_0$ for all $t \geq 0$ and N sufficiently large.

B. *Some Remarks on the Assumptions and the Proofs of our Results*

As noted in the preceding subsection, in case $\beta \in (0, 1)$ any fixed particle interacts with many ($\approx N \operatorname{vol}(B(\chi_N^{-d}, \cdot)) = N^{1-\beta}$) different neighbours. The strength of the interaction with any fixed neighbour is of order $(1/N) V_N(0) \approx N^{\beta - 1}$, which gets small as $N \to \infty$. On the other hand, for $\beta = 1$ both the number of neighbours, which have at some fixed instant of time an appreciable effect on the motion of a fixed particle, and the size of this effect remain of finite order as $N \to \infty$. This entails that in the case $\beta \in (0, 1)$ the individual (microscopic) behaviour of the particles stays in the background, such that we can concentrate on the bulk (macroscopic) behaviour of the whole population. This is reflected in the L^2-techniques in the proof of Theorem (2.21) and in the assumptions on V_1 and W_1, which are adapted to this L^2-approach. In the case of convergence to such solutions of (2.12), which have compact support for all $t \geqslant 0$, the individual behaviour of those particles, which stay near the boundary of the support of $p(\cdot, t)$, gets more important. Therefore it is not surprising that we here need a new assumption on V_1, namely (2.8), which better allows us to control the microscopic behaviour of those particles.

In the hydrodynamic case $\beta = 1$ we need a good knowledge on the local geometry of the particle configurations. We obtain this knowledge by choosing *regular initial configurations* (3.3)–(3.6) and a nice interaction potential V_1 (cf. (3.1), (3.2)), which allows us to show that the particle configurations stay *regular* at any time $t \geqslant 0$ (cf. Section 6.A). Next, we can deduce the validity of the property of *local equilibrium* (cf. (1.11) or more precisely Section 6.B). This property means that for large N most of the time locally the particle configuration looks like an invariant configuration of the corresponding infinite particle system (cf. [12, 20]). Those invariant configurations are characterized in [10] through equal spacings between the particles.

As indicated in the Introduction (cf. (1.9)–(1.12)), the validity of local equilibrium is decisive for the identification of the limit dynamics. Let us mention now that a large amount of the technical difficulties in the proof of Theorem (3.8) is induced by boundary effects. In those regions, where the *density* of the population of the X_N^k is too low, i.e., near the free boundary of the support of the solution of (1.14), the particles don't feel much interaction and therefore local equilibrium cannot be established.

During the proof of Theorem (2.21) one may wonder where the assumption $\beta < 1$ is needed. We shall indicate now that it is assumption (2.22), which can be satisfied if and only if $\beta < 1$.

A natural choice of the initial configuration $X_N(0)$ is to assume that $X_N^k(0)$, $k = 1, ..., N$, are independent, identically distributed random variables, whose distribution has density p_0. Then (we assume $G[V_1] = 1$)

$$\begin{aligned}
&E[\|h_N(\cdot, 0) - p_0\|_2^2] \\
&\quad = E[\|h_N(\cdot, 0)\|_2^2] - 2E[\langle h_N(\cdot, 0), p_0\rangle] + \|p_0\|_2^2 \\
&\quad = \frac{1}{N^2} \sum_{k,l=1}^{N} E[V_N(X_N^k(0) - X_N^l(0))] \\
&\qquad - \frac{2}{N} \sum_{k=1}^{N} E[(p_0 * W_N^-)(X_N^k(0))] + \|p_0\|_2^2 \qquad \text{(by (2.2))} \\
&\quad = \frac{1}{N^2} \sum_{\substack{k,l=1 \\ k \neq l}}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_0(x)\, p_0(y)\, V_N(x-y)\, dx\, dy + \frac{1}{N} V_N(0) \\
&\qquad - \frac{2}{N} \sum_{k=1}^{N} \int_{\mathbb{R}^d} (p_0 * W_N^-)(x)\, p_0(x)\, dx + \|p_0\|_2^2 \\
&\quad = \frac{1}{N} V_N(0) - \frac{1}{N} \|p_0 * W_N\|_2^2 + \|p_0 - p_0 * W_N\|_2^2.
\end{aligned}$$

Hence, since $(1/N)\, V_N(0) = N^{\beta-1} V_1(0)$,

$$\lim_{N \to \infty} E[\|h_N(\cdot, 0) - p_0\|_2^2] = 0, \qquad \text{if} \quad \beta \in (0, 1). \tag{4.2}$$

An application of the zero-one law even shows

$$\lim_{N \to \infty} \|h_N(\cdot, 0) - p_0\|_2^2 = 0 \qquad \text{almost surely,}$$

i.e., (2.22) is true, if $\beta \in (0, 1)$, for almost all sequences of initial configurations $X_N(0) = (1/N) \sum_{k=1}^{N} \delta_{X_N^k(0)}$, $N = 1, 2, \ldots$.

On the other hand, for $\beta = 1$ the convergence (4.2) cannot hold any more, since $V_1(0) = 0$ is excluded by (2.2).

Even if we choose the empirical distribution $X_N(0)$ in a deterministic way, (2.22) cannot be satisfied for $\beta = 1$. For example, a "best approximation" of the function $\mathbb{1}_{[0,1]}$ is provided in this situation by putting the particles at positions k/N, $k = 0, \ldots, N-1$. Then we have

$$h_N(x, 0) = \sum_{k=0}^{N-1} W_1(Nx - k),$$

and therefore for N sufficiently large

$$\begin{aligned}\|h_N(\cdot, 0) - \mathbb{1}_{[0,1]}\|_2^2 &\geqslant \int_{1/4}^{3/4} (h_N(x, 0) - 1)^2 \, dx \\ &= \int_{1/4}^{3/4} (\theta(Nx) - 1)^2 \, dx + o(1) \\ &\qquad \left(\theta(x) = \sum_{k=-\infty}^{\infty} W_1(x-k)\right) \\ &= \frac{1}{N} \int_{N/4}^{3N/4} (\theta(x) - 1)^2 \, dx + o(1) \geqslant C > 0,\end{aligned}$$

since θ has period 1 and is not identically 1.

These arguments imply that we cannot expect a L^2-convergence result like (2.23) in the case $\beta = 1$.

C. *Related Work*

As far as the case $\beta \in (0, 1)$ is concerned this work has to be seen in relation to several papers [11, 15–17] where the convergence of the empirical processes of systems of *moderately interacting* stochastic processes to the solution of certain reaction-diffusion equations is studied. In all these cases a Brownian motion-part in the dynamics of the individual stochastic processes is added. Correspondingly, its deterministic counterpart, namely a uniformly elliptic second order differential operator, occurred in the PDEs describing the limit processes in these situations. Therefore, one has a certain regularizing effect, which facilitates the proof of the convergence results considerably.

A natural question, which arises in this context, asks for the validity of such convergence results, if this regularizing effect is missing, i.e., if one has purely deterministic interacting processes. The model (1.1), which has been chosen to investigate this problem, is closely related to [15, 16], where we studied systems of $\mathbb{R}^d$-valued interacting diffusion processes $\bar{X}_N^k(t)$, $k = 1, ..., N$, satisfying the stochastic differential equations

$$d\bar{X}_N^k(t) = -\frac{1}{N} \sum_{l=1}^{N} \nabla V_N(\bar{X}_N^k(t) - \bar{X}_N^l(t)) \, dt + dW^k(t), \qquad k = 1, ..., N, \tag{4.3}$$

where W^k, $k = 1, 2, ...$, are independent $\mathbb{R}^d$-valued standard Brownian motions. In this case we derived as limit dynamics the viscous Boussinesq equation

$$\frac{\partial}{\partial t} \bar{p}(x, t) = G[V_1] \nabla \cdot (\bar{p}(x, t) \nabla \bar{p}(x, t)) + \frac{1}{2} \Delta \bar{p}(x, t).$$

Unfortunately the arguments in [15, 16] rely heavily on the presence of the Brownian motion in (4.3). Hence, we have to do additional work to cover the case (1.1). In particular, we have to overcome problems caused by the nonregularity of the solution of (1.4), resp. its weak version (2.12), at the free boundary of its support.

Another derivation of the porous medium equation for the *macroscopic description* of a many-particle system has been accomplished in the recent paper [22]. In this approach for any $\alpha > 1$ the microscopic dynamics of the particles is given in terms of coupled stochastic jump processes in discrete time on a lattice in $\mathbb{R}^d$. By adjusting the space-time discretization in a way corresponding to our scaling with $\beta < 1$ the convergence of the empirical processes to the solution of

$$\frac{\partial}{\partial t} p(x, t) = \frac{1}{2} \Delta p^{\alpha}(x, t)$$

is shown.

As far as our paper describes the derivation of the porous medium equation from an underlying deterministic dynamics of individuals, who avoid crowding, it is related to [8], where this partial differential equation is used as a model for the description of the time evolution of the density of a population of individuals exhibiting *directed motion* in space.

One of the main ingredients of the usual phenomenological derivation of the porous medium equation as a model for the flow of an ideal gas in a homogeneous porous medium (cf. [13]) is *Darcy's law*

$$\tau v = -\mu \nabla p \tag{4.4}$$

(v velocity, τ viscosity of the gas, μ permeability of the medium, p pressure). In case of an isothermic flow additionally one has the *equation of state*

$$p = nRT/w \tag{4.5}$$

(n density of the gas, R gas constant per mole, w molecular weight, T temperature). Equations (4.4), (4.5) imply

$$v = -\frac{\mu R}{\tau w} T \nabla n.$$

This equation can be considered as a phenomenological counterpart of our "microscopic" dynamics (1.7), at least for $\beta \in (0, 1)$, where, as discussed in Subsection 4.A, $g_N(\cdot, t)$ represents the particle density. Hence, Boussinesq's equation appears quite naturally as limit dynamics.

As far as the case $\beta = 1$ is concerned, there are connections to [20], where, as mentioned above, the system (4.1) with additional Brownian motion in the dynamics of the Y_N^k is discussed. In particular, its behaviour in the hydrodynamic limit is studied.

Another model, which is closely related to (4.1), is investigated in [12], where the summation on the right side of (4.1) is replaced by a summation over the neighbours $k-1$, $k+1$. Hence, the dynamics in [12] is simpler than ours. However, there are allowed quite general initial conditions, and restrictions like (3.3)–(3.6) are not needed. In that situation too a generalized porous medium equation is obtained in the limit $N \to \infty$.

5. Proof of Theorem (2.21)

Without restricting generality we assume in this section $\int_{\mathbb{R}^d} W_1(x)\,dx = 1$. In particular, this implies $G[V_1]=1$, $h_N(\cdot, t) = (X_N(t) * W_N)(\cdot)$, and $g_N(\cdot, t) = (h_N(\cdot, t) * W_N^-)(\cdot)$, where $W_N^-(x) = \chi_N^d W_1^-(\chi_N x)$.

To simplify the calculations and the notation in this proof we use the representation (1.7) for the dynamics (1.1); i.e., we behave as if $V_1 \in C_b^1(\mathbb{R}^d)$ with $\nabla V_1(0)=0$. In this case we have to prove the version (2.26) of (2.23), cf. Remark (2.25). Moreover, we ignore the fact that in general $\nabla h_N(\cdot, t)$ (N, t fixed) is well defined only as a generalized function, since W_1 does not need to be differentiable. We shall transform (usually by integration by parts) any term containing $\nabla h_N(\cdot, t)$ into an equivalent expression involving $h_N(\cdot, t)$ and some other well defined functions. For this new expression suitable estimates are obtained. This formal procedure seems to be inconsistent in particular with assumption (2.8), which implies that V_1' has to be discontinuous in 0. However, it may easily be justified, e.g., in the case $d=1$, (2.2)–(2.9), (P2), essentially by using (2.11) as a priori estimate, and then approximating for any fixed $N \in \mathbb{N}$ the dynamics (1.1) by similar systems of differential equations involving smooth $W_{1,\varepsilon}$, such that $\lim_{\varepsilon \to 0} \sup\{|V_N^{(k)}(x) - V_{N,\varepsilon}^{(k)}(x)| : |x| \geqslant d_N e^{-N}\} = 0$, $k=0, 1$.

A. *Preliminary Calculations*

We obtain by (1.7), (2.2) for $t \geqslant 0$

$$\begin{aligned} \|h_N(\cdot, t)\|_2^2 &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} X_N(t)(dx)\, W_N(z-x) \right|^2 dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} X_N(t)(dx)\, X_N(t)(dy)\, V_N(x-y) \\ &= \|h_N(\cdot, 0)\|_2^2 - 2 \int_0^t \langle X_N(s), \nabla g_N(\cdot, s)\, \nabla g_N(\cdot, s) \rangle\, ds. \quad (5.1) \end{aligned}$$

Next by (1.8), (2.12), (2.16), (2.19), (P1),

$$\begin{aligned} \langle h_N(\cdot, t), p(\cdot, t) \rangle &= \langle X_N(t), p(\cdot, t) * W_N^- \rangle \\ &= \langle h_N(\cdot, 0), p_0 \rangle \\ &\quad - \int_0^t \{ \langle X_N(s), \nabla g_N(\cdot, s)\, \nabla(p(\cdot, s) * W_N^-) \rangle \\ &\quad + \langle p(\cdot, s), \nabla p(\cdot, s)\, \nabla h_N(\cdot, s) \rangle \}\, ds, \quad (5.2) \end{aligned}$$

and

$$\|p(\cdot,t)\|_2^2=\|p_0\|_2^2-2\int_0^t \langle p(\cdot,s),\nabla p(\cdot,s)\,\nabla p(\cdot,s)\rangle\,ds. \tag{5.3}$$

Equations (5.1)–(5.3) imply

$$\begin{aligned}
&\|h_N(\cdot,t)-p(\cdot,t)\|_2^2\\
&\quad=\|h_N(\cdot,t)\|_2^2-2\langle h_N(\cdot,t),p(\cdot,t)\rangle+\|p(\cdot,t)\|_2^2\\
&\quad=\|h_N(\cdot,0)-p_0\|_2^2-2\int_0^t \{\langle X_N(s),|\nabla g_N(\cdot,s)|^2\rangle\\
&\qquad-2\langle X_N(s),\nabla g_N(\cdot,s)\,\nabla p(\cdot,s)\rangle+\langle X_N(s),|\nabla p(\cdot,s)|^2\rangle\}\,ds\\
&\qquad+2\int_0^t \{\langle X_N(s),|\nabla p(\cdot,s)|^2\rangle-\langle X_N(s),\nabla g_N(\cdot,s)\,\nabla p(\cdot,s)\rangle\\
&\qquad-\langle p(\cdot,s),|\nabla p(\cdot,s)|^2\rangle+\langle p(\cdot,s),\nabla g_N(\cdot,s)\,\nabla p(\cdot,s)\rangle\}\,ds\\
&\qquad+2\int_0^t \{\langle X_N(s),\nabla g_N(\cdot,s)(-\nabla p(\cdot,s)+\nabla p(\cdot,s)*W_N^-)\rangle\\
&\qquad+\langle p(\cdot,s),\nabla p(\cdot,s)(-\nabla g_N(\cdot,s)+\nabla h_N(\cdot,s))\rangle\}\,ds\\
&\quad=\|h_N(\cdot,0)-p_0\|_2^2-2\int_0^t \langle X_N(s),|\nabla g_N(\cdot,s)-\nabla p(\cdot,s)|^2\rangle\,ds\\
&\qquad+2\int_0^t \langle X_N(s)-p(\cdot,s),\nabla p(\cdot,s)(\nabla p(\cdot,s)*W_N^- -\nabla g_N(\cdot,s))\rangle\,ds\\
&\qquad+2\int_0^t \{\langle X_N(s),(\nabla g_N(\cdot,s)-\nabla p(\cdot,s))(\nabla p(\cdot,s)*W_N^- -\nabla p(\cdot,s))\rangle\\
&\qquad+\langle p(\cdot,s),\nabla p(\cdot,s)(\nabla h_N(\cdot,s)\\
&\qquad\qquad-\nabla g_N(\cdot,s)-\nabla p(\cdot,s)+\nabla p(\cdot,s)*W_N^-)\rangle\}\,ds.
\end{aligned} \tag{5.4}$$

Formally, the last integral on the right side of (5.4) can be neglected in the limit $N\to\infty$, whereas the third term is of order

$$\begin{aligned}
&\approx 2\int_0^t \langle h_N(\cdot,s)-p(\cdot,s),\nabla p(\cdot,s)(\nabla p(\cdot,s)-\nabla h_N(\cdot,s))\rangle\,ds\\
&\qquad=\int_0^t \langle (h_N(\cdot,s)-p(\cdot,s))^2,\Delta p(\cdot,s)\rangle\,ds.
\end{aligned}$$

If (P1) is valid, the last expression is less than $C\int_0^t \|h_N(\cdot,s)-p(\cdot,s)\|_2^2\,ds$, and (2.26) "follows" after an application of Gronwall's Lemma.

In the remaining subsections we have to make this treatment of the right side of (5.4) precise.

B. *Convergence to Classical Solutions of Boussinesq's Equation*

For simplicity of notation we assume from now on $d=1$. In the case of validity of (P1) the arguments below are easily generalized to the case $d > 1$. The only difference appears in (5.12), where for $d > 1$ we have to take into account higher terms of the Taylor expansion of $p(\cdot, s)$.

We have to find useful upper bounds for the third and fourth integral on the right side of (5.4). For that purpose we need:

LEMMA (5.5). (a) *Let $f \in C_b^1(\mathbb{R})$. Then for any $x \in \mathbb{R}$*

$$|f(x) - (f * W_N)(x)| \leqslant C\chi_N^{-1} \|\nabla f\|_\infty. \tag{5.6}$$

(b) *Let $f, \nabla f \in L^2(\mathbb{R})$. Then*

$$\|f - f * W_N\|_2^2 \leqslant C\chi_N^{-2} \|\nabla f\|_2^2. \tag{5.7}$$

Similar estimates are true, if we replace W_N by W_N^- or V_N.

Proof.

(a)
$$\begin{aligned}
|f(x) - (f * W_N)(x)| &= \left| \int_{\mathbb{R}} (f(x) - f(x-y))\, W_N(y)\, dy \right| \\
&\qquad \left(\text{since} \int_{\mathbb{R}} W_N(x)\, dx = 1 \right) \\
&\leqslant \int_{\mathbb{R}} \|\nabla f\|_\infty\, |y|\, |W_N(y)|\, dy \\
&= \chi_N^{-1} \|\nabla f\|_\infty \int_{\mathbb{R}} \chi_N\, |y|\, |W_N(y)|\, dy \\
&= \chi_N^{-1} \|\nabla f\|_\infty \int_{\mathbb{R}} |x|\, |W_1(x)|\, dx \\
&= C\chi_N^{-1} \|\nabla f\|_\infty \qquad (\text{by } (2.4)).
\end{aligned} \tag{5.8}$$

(b)
$$\begin{aligned}
\|f - f * W_N\|_2^2 &= \int_{\mathbb{R}} |\tilde{f}(\mu)|^2\, |1 - (2\pi)^{1/2}\, \widetilde{W_N}(\mu)|^2\, d\mu \\
&= \int_{\mathbb{R}} |\tilde{f}(\mu)|^2\, |1 - (2\pi)^{1/2}\, \widetilde{W_1}(\mu/\chi_N)|^2\, d\mu \\
&= \int_{\mathbb{R}} |\tilde{f}(\mu)|^2 \frac{|1 - (2\pi)^{1/2}\, \widetilde{W_1}(\mu/\chi_N)|^2}{|\mu/\chi_N|^2} \left| \frac{\mu}{\chi_N} \right|^2 d\mu \\
&\leqslant \chi_N^{-2}(2\pi)\, \|\nabla \widetilde{W_1}\|_\infty^2 \int_{\mathbb{R}} |\tilde{f}(\mu)|^2\, \mu^2\, d\mu \\
&\qquad \left(\text{since } \widetilde{W_1}(0) = (2\pi)^{-1/2} \int_{\mathbb{R}} W_1(x)\, dx = (2\pi)^{-1/2} \right) \\
&\leqslant C\chi_N^{-2} \|\nabla f\|_2^2 \qquad (\text{by } (2.3)).
\end{aligned}$$

Now, we have

$$
\begin{aligned}
&|\langle X_N(s), (g'_N(\cdot, s) - p'(\cdot, s))(p'(\cdot, s) * W_N^- - p'(\cdot, s))\rangle| \\
&\quad \leqslant \langle X_N(s), |g'_N(\cdot, s) - p'(\cdot, s)|^2\rangle^{1/2} \\
&\qquad \times \langle X_N(s), |p'(\cdot, s) * W_N^- - p'(\cdot, s)|^2\rangle^{1/2} \\
&\quad \leqslant \frac{1}{2}\langle X_N(s), |g'_N(\cdot, s) - p'(\cdot, s)|^2\rangle \\
&\qquad + \frac{1}{2}\langle X_N(s), |p'(\cdot, s) * W_N^- - p'(\cdot, s)|^2\rangle \qquad (5.9) \\
&\quad \leqslant \frac{1}{2}\langle X_N(s), |g'_N(\cdot, s) - p'(\cdot, s)|^2\rangle \\
&\qquad + C\,\|p''(\cdot, s)\|_\infty^2\, \chi_N^{-2} \qquad \text{(by (5.6))}, \qquad (5.10)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\langle p(\cdot, s), p'(\cdot, s)(h'_N(\cdot, s) - g'_N(\cdot, s) - p'(\cdot, s) + p'(\cdot, s) * W_N^-)\rangle| \\
&\quad = |\langle p(\cdot, s), p'(\cdot, s)(h'_N(\cdot, s) - p'(\cdot, s) - (h'_N(\cdot, s) - p'(\cdot, s)) * W_N^-)\rangle| \\
&\quad = |\langle p(\cdot, s)\, p'(\cdot, s) - (p(\cdot, s)\, p'(\cdot, s)) * W_N, h'_N(\cdot, s) - p'(\cdot, s)\rangle| \\
&\quad \leqslant \|h_N(\cdot, s) - p(\cdot, s)\|_2^2 + C\chi_N^{-2}\, \|(p^2(\cdot, s))'''\|_2^2 \qquad \text{(by (5.7))} \\
&\quad \leqslant \|h_N(\cdot, s) - p(\cdot, s)\|_2^2 \\
&\qquad + C\chi_N^{-2}(\|p''(\cdot, s)\, p'(\cdot, s)\|_2^2 + \|p(\cdot, s)\, p'''(\cdot, s)\|_2^2) \\
&\quad \leqslant \|h_N(\cdot, s) - p(\cdot, s)\|_2^2 + C\chi_N^{-2}(\|p''(\cdot, s)\|_\infty^2 + \|p'''(\cdot, s)\|_\infty^2) \\
&\qquad\quad \text{(by (2.16), (2.19), (2.20))}. \qquad (5.11)
\end{aligned}
$$

For the integrand of the third expression on the right side of (5.4) we obtain

$$
\begin{aligned}
&\langle X_N(s) - p(\cdot, s), p'(\cdot, s)(p'(\cdot, s) * W_N^- - g'_N(\cdot, s))\rangle \\
&\quad = \langle X_N(s) - p(\cdot, s), p'(\cdot, s)(p(\cdot, s) - h_N(\cdot, s)) * (W_N^-)'\rangle \\
&\quad = \Big\langle X_N(s) - p(\cdot, s), \int_{\mathbb{R}} (W_N^-)'(u)\Big[p'(\cdot - u, s) + p''(\cdot - u, s)\, u \\
&\qquad\quad + p'''(\cdot - u + \theta(\cdot, u, s)\, u, s)\frac{u^2}{2}\Big](p(\cdot - u, s) - h_N(\cdot - u, s))\, du\Big\rangle \\
&\quad = R^*_{N,0}(s) + R^*_{N,1}(s) + R^{**}_N(s), \qquad (5.12)
\end{aligned}
$$

where $|\theta(\cdot, \cdot, \cdot)| \leqslant 1$,

$$R^*_{N,k}(s) = \langle (X_N(s) - p(\cdot, s)) * U^{[1]}_{N,k;1}, p^{(k+1)}(\cdot, s) \times (p(\cdot, s) - h_N(\cdot, s)) \rangle, \qquad k = 0, 1, \tag{5.13}$$

with

$$U^{[1]}_{N,k;1}(x) = \chi_N^{-k+2} U^{[1]}_{1,k;1}(\chi_N x), \qquad k = 0, 1, 2, \tag{5.14}$$

and

$$R^{**}_N(s) = \left\langle X_N(s) - p(\cdot, s), \int_{\mathbb{R}} (W_N^-)'(u)\, p'''(\cdot - u + \theta(\cdot, u, s)\, u, s) \frac{u^2}{2} \times (p(\cdot - u, s) - h_N(\cdot - u, s))\, du \right\rangle. \tag{5.15}$$

We can estimate the first term in (5.12) as

$$\begin{aligned} |R^*_{N,0}(s)| &= |\langle (X_N(s) - p(\cdot, s)) * W'_N, p'(\cdot, s)(p(\cdot, s) - h_N(\cdot, s)) \rangle| \\ &\leqslant |\langle h'_N(\cdot, s) - p'(\cdot, s), p'(\cdot, s)(p(\cdot, s) - h_N(\cdot, s)) \rangle| \\ &\quad + |\langle p'(\cdot, s) - p'(\cdot, s) * W_N, p'(\cdot, s)(p(\cdot, s) - h_N(\cdot, s)) \rangle| \qquad (5.16) \\ &\leqslant C(\|h_N(\cdot, s) - p(\cdot, s)\|_2^2 \, \|p''(\cdot, s)\|_\infty \\ &\quad + \|h_N(\cdot, s) - p(\cdot, s)\|_2 \, \|p'(\cdot, s)\|_2 \, \|p''(\cdot, s)\|_\infty \, \chi_N^{-1}) \\ &\quad \text{(by (5.6))} \\ &\leqslant C(\|h_N(\cdot, s) - p(\cdot, s)\|_2^2 + \chi_N^{-2}) \, \|p''(\cdot, s)\|_\infty \\ &\quad \text{(by (2.19), (2.20))}. \end{aligned} \tag{5.17}$$

Note that the estimation of the first term on the right side of (5.16) was the main reason for introducing the assumption (P1). Essentially, we could not find any bound for $(h'_N(\cdot, s) - p'(\cdot, s))\ p'(\cdot, s)(h_N(\cdot, s) - p(\cdot, s))$ in that region of $\mathbb{R}$, where $p''(\cdot, s)$ is singular.

Furthermore, for $k = 1$

$$\begin{aligned} |R^*_{N,1}(s)| &= |\langle (X_N(s) - p(\cdot, s)) * U^{[1]}_{N,1;1}, p''(\cdot, s)(p(\cdot, s) - h_N(\cdot, s)) \rangle| \\ &\leqslant \|p''(\cdot, s)\|_\infty \, \|(X_N(s) - p(\cdot, s)) * U^{[1]}_{N,1;1}\|_2 \\ &\quad \times \|h_N(\cdot, s) - p(\cdot, s)\|_2. \end{aligned} \tag{5.18}$$

Relations (2.6), (5.14) imply

$$|\widetilde{U^{[1]}_{N,1;1}}(\mu)| = |\widetilde{U^{[1]}_{N,1;1}}(\mu/\chi_N)| \leqslant C\, |\widetilde{W_N}(\mu)|. \tag{5.19}$$

Hence

$$
\begin{aligned}
&\|(X_N(s) - p(\cdot, s)) * U^{[1]}_{N,1;1}\|_2^2 \\
&\quad = C \int_{\mathbb{R}} |\widetilde{X_N(s)}(\mu) - \tilde{p}(\mu, s)|^2 \, |\widetilde{U^{[1]}_{N,1;1}}(\mu)|^2 \, d\mu \\
&\quad \leqslant C \int_{\mathbb{R}} |\widetilde{X_N(s)}(\mu) - \tilde{p}(\mu, s)|^2 \, |\widetilde{W_N}(\mu)|^2 \, d\mu \\
&\quad = C \, \|h_N(\cdot, s) - p(\cdot, s) * W_N\|_2^2 \\
&\quad \leqslant C(\|h_N(\cdot, s) - p(\cdot, s)\|_2^2 + \|p(\cdot, s) - p(\cdot, s) * W_N\|_2^2) \\
&\quad \leqslant C(\|h_N(\cdot, s) - p(\cdot, s)\|_2^2 + \chi_N^{-2}) \\
&\qquad\qquad \text{(by (2.19), (2.20), (5.7)).}
\end{aligned} \tag{5.20}
$$

Estimates (5.18)–(5.20) imply

$$
\begin{aligned}
|R^*_{N,1}(s)| &\leqslant C \, \|p''(\cdot, s)\|_\infty \, (\|h_N(\cdot, s) - p(\cdot, s)\|_2 + \chi_N^{-1}) \\
&\qquad \times \|h_N(\cdot, s) - p(\cdot, s)\|_2 \\
&\leqslant C \, \|p''(\cdot, s)\|_\infty \, (\|h_N(\cdot, s) - p(\cdot, s)\|_2^2 + \chi_N^{-2}).
\end{aligned} \tag{5.21}
$$

Next, we obtain

$$
\begin{aligned}
|R^{**}_N(s)| &\leqslant \Big\langle X_N(s) + p(\cdot, s), \int_{\mathbb{R}} |(W_N^-)'(u)| \\
&\qquad \times \frac{u^2}{2} \, |p(\cdot - u, s) - h_N(\cdot - u, s)| \, du \Big\rangle \, \|p'''(\cdot, s)\|_\infty \\
&= \langle X_N(s) * |U^{[1]}_{N,2;1}| + p(\cdot, s) * |U^{[1]}_{N,2;1}|, |p(\cdot, s) - h_N(\cdot, s)| \rangle \\
&\quad \times \|p'''(\cdot, s)\|_\infty.
\end{aligned} \tag{5.22}
$$

We now observe

$$
\begin{aligned}
&\langle X_N(s) * |U^{[1]}_{N,2;1}|, |p(\cdot, s) - h_N(\cdot, s)| \rangle \\
&\quad = \int_{\mathbb{R}} X_N(s)(dx) \int_{\mathbb{R}} dy \, |U^{[1]}_{N,2;1}(y - x)| \, |p(y, s) - h_N(y, s)| \\
&\quad \leqslant C \int_{\mathbb{R}} X_N(s)(dx) \int_{\mathbb{R}} dy \, \frac{1}{1 + |\chi_N(y - x)|} \, |p(y, s) - h_N(y, s)| \\
&\qquad \text{(by (2.7), (5.14))} \\
&\quad \leqslant C \int_{\mathbb{R}} X_N(s)(dx) \left\| \frac{1}{1 + |\chi_N(\cdot - x)|} \right\|_2 \, \|p(\cdot, s) - h_N(\cdot, s)\|_2 \\
&\quad \leqslant C \chi_N^{-1/2} \, \|p(\cdot, s) - h_N(\cdot, s)\|_2,
\end{aligned} \tag{5.23}
$$

and similarly,

$$\langle p(\cdot, s) * |U_{N,2;1}^{[1]}|, |p(\cdot, s) - h_N(\cdot, s)| \rangle$$
$$\leqslant C\chi_N^{-1/2} \|p(\cdot, s) - h_N(\cdot, s)\|_2. \tag{5.24}$$

Hence, by (5.22)–(5.24)

$$|R_N^{**}(s)| \leqslant C\chi_N^{-1/2} \|p(\cdot, s) - h_N(\cdot, s)\|_2 \, \|p'''(\cdot, s)\|_\infty$$
$$\leqslant C(\|p(\cdot, s) - h_N(\cdot, s)\|_2^2 + \chi_N^{-1} \|p'''(\cdot, s)\|_\infty^2). \tag{5.25}$$

Relations (5.12), (5.17), (5.21), (5.25) imply

$$|\langle X_N(s) - p(\cdot, s), p'(\cdot, s)(p'(\cdot, s) * W_N^- - g'_N(\cdot, s))\rangle|$$
$$\leqslant C((\|h_N(\cdot, s) - p(\cdot, s)\|_2^2 + \chi_N^{-2})\,(\|p''(\cdot, s)\|_\infty + 1)$$
$$+ \chi_N^{-1} \|p'''(\cdot, s)\|_\infty^2). \tag{5.26}$$

Now we can finish the proof of Theorem (2.21) fairly quickly. From (5.4), (5.10), (5.11), (5.26) we conclude

$$\|h_N(\cdot, t) - p(\cdot, t)\|_2^2$$
$$\leqslant \|h_N(\cdot, 0) - p_0\|_2^2 - \int_0^t \langle X_N(s), |g'_N(\cdot, s) - p'(\cdot, s)|^2\rangle \, ds$$
$$+ C\int_0^t [(\|h_N(\cdot, s) - p(\cdot, s)\|_2^2 + \chi_N^{-2})(1 + \|p''(\cdot, s)\|_\infty^2)$$
$$+ \chi_N^{-1} \|p'''(\cdot, s)\|_\infty^2] \, ds.$$

Hence, by Gronwall's Lemma for any $t \in [0, T]$

$$\|h_N(\cdot, t) - p(\cdot, t)\|_2^2 + \int_0^t \langle X_N(s), |g'_N(\cdot, s) - p'(\cdot, s)|^2\rangle \, ds$$
$$\leqslant [\|h_N(\cdot, 0) - p_0\|_2^2 + CT(\chi_N^{-2}(1 + \|p''\|_\infty^2) + \chi_N^{-1} \|p'''\|_\infty^2)]$$
$$\times \exp(CT(1 + \|p''\|_\infty^2)). \tag{5.27}$$

By (2.22) we have finished now the proof of Theorem (2.21)(i) in the particular case that (P1) is satisfied.

C. *The Derivation of* (2.24) *from* (2.23)

First, we note

$$|\langle X_N(t) - p(\cdot, t), f\rangle|$$
$$\leqslant |\langle h_N(\cdot, t) - p(\cdot, t), f\rangle| + \langle X_N(t), |f - f * W_N^-|\rangle,$$

where by (5.6) the second term on the right side tends to 0 as $N \to \infty$ uniformly in $t \in [0, T]$ and $f \in \mathscr{H}_1 = \{f \in C_b^1(\mathbb{R}) : \max\{\|f\|_\infty, \|f'\|_\infty\} \leqslant 1\}$. Next, we choose for fixed $\varepsilon > 0$ a ball $B_R = \{x \in \mathbb{R} : |x| \leqslant R\}$, such that

$$\sup_{t \leqslant T} \langle p(\cdot, t), \mathbb{1}_{\mathbb{R} \setminus B_R} \rangle \leqslant \varepsilon,$$

and a positive function $\theta \in \mathscr{H}_1$ with $\theta(x) = 1$ for $|x| \leqslant R$ and $\theta(x) = 0$ for $|x| \geqslant R + 2$.

We can write any $f \in \mathscr{H}_1$ as $f = f_R + \hat{f}_R$, where f_R, $\hat{f}_R \in \mathscr{H}_1$ with $\operatorname{supp}(f_R) \subseteq B_{R+4}$, $\operatorname{supp}(\hat{f}_R) \subseteq \mathbb{R} \setminus B_{R+2}$.

Then we observe

$$\begin{aligned}
&|\langle h_N(\cdot, t) - p(\cdot, t), f \rangle| \\
&\qquad \leqslant |\langle h_N(\cdot, t) - p(\cdot, t), f_R \rangle| + |\langle h_N(\cdot, t), \hat{f}_R \rangle| + \langle p(\cdot, t), \mathbb{1}_{\mathbb{R} \setminus B_R} \rangle, \\
&\lim_{N \to \infty} \sup_{t \leqslant T, f \in \mathscr{H}_1} |\langle X_N(t) - h_N(\cdot, t), \hat{f}_R \rangle| = 0 \qquad \text{(by (5.6))}, \\
&|\langle X_N(t), \hat{f}_R \rangle| \\
&\qquad \leqslant \langle X_N(t), \mathbb{1}_{\mathbb{R} \setminus B_{R+2}} \rangle \leqslant \langle X_N(t), 1 - \theta \rangle \\
&\qquad = \langle p(\cdot, t) - h_N(\cdot, t), \theta \rangle + \langle h_N(\cdot, t) - X_N(t), \theta \rangle \\
&\qquad\quad + \langle p(\cdot, t), 1 - \theta \rangle, \\
&\langle p(\cdot, t), 1 - \theta \rangle \leqslant \langle p(\cdot, t), \mathbb{1}_{\mathbb{R} \setminus B_R} \rangle,
\end{aligned}$$

and

$$\lim_{N \to \infty} \sup_{t \leqslant T} |\langle X_N(t) - h_N(\cdot, t), \theta \rangle| = 0 \qquad \text{(by (5.6))}.$$

These estimates together with (2.23) imply

$$\begin{aligned}
&\lim_{N \to \infty} \sup_{t \leqslant T} \|X_N(t) - p(\cdot, t)\|_* \\
&\qquad \leqslant C\Big(\lim_{N \to \infty} \sup_{t \leqslant T, f \in \mathscr{H}_1} \|h_N(\cdot, t) - p(\cdot, t)\|_2 \,(\|f_R\|_2 + \|\theta\|_2) + \varepsilon\Big) = C\varepsilon.
\end{aligned}$$

Since ε was arbitrary, (2.24) is proved.

D. *An Auxiliary Result*

From now on we restrict our considerations to the case $d = 1$, and assume the validity of (1.1)–(1.3), (2.2)–(2.9), (2.14), (2.18), (2.22) and (P2).

In an intermediate step we need the measure-valued processes $X_{N,a}$, which are obtained from X_N by absorbing the processes $X_N^k(t)$ at the boundary ∂Q_t of $Q_t = \operatorname{supp}(p(\cdot, t))$.

More precisely, we define

$$X_{N,a}(t)=\frac{1}{N}\sum_{k\in L_N(t)}\delta_{X^k_{N,a}(t)},$$

where $X^k_{N,a}(\cdot)=X^k_{N,a}(t)$, $0\leqslant t\leqslant t^k_N$, solves

$$\frac{d}{dt}X^k_{N,a}(t)=-\frac{1}{N}\sum_{\substack{m\in L_N(t)\\ m\neq k}}V'_N(X^k_{N,a}(t)-X^m_{N,a}(t)),$$

$$X^k_{N,a}(0)=X^k_N(0),$$

with

$$t^k_N=\inf\{t\geqslant 0:X^k_{N,a}(t)\notin \operatorname{int}(Q_t)\}$$

and

$$L_N(t)=\{k\in\{1,...,N\}:t^k_N\geqslant t\}.$$

For $h_{N,a}(\cdot,t)=(X_{N,a}(t)*W_N)(\cdot)$ and $g_{N,a}(\cdot,t)=(X_{N,a}(t)*V_N)(\cdot)$ we obtain an analogue of (2.23), resp. (2.26):

PROPOSITION (5.28). *Assume* (1.1)–(1.3), (2.2)–(2.9), (2.14), (2.18), (2.22), (P2). *Then*

$$\lim_{N\to\infty}\left\{\sup_{t\leqslant T}\|h_{N,a}(\cdot,t)-p(\cdot,t)\|_2^2\right.$$
$$\left.+\int_0^T\langle X_{N,a}(s),|g'_{N,a}(\cdot,s)-p'(\cdot,s)|^2\rangle\,ds\right\}=0. \tag{5.29}$$

Proof. The main difficulty in the proof of this proposition is the estimation of the "distance" between $h_{N,a}(\cdot,t)$ and $p(\cdot,t)$ in the region near ∂Q_t, i.e., where $p(\cdot,t)$ is not smooth. In the interior of Q_t or $\mathbb{R}\backslash Q_t$ we can apply techniques still used in Subsection 5.B.

First, let us fix a sequence $\varepsilon_1>\varepsilon_2>\cdots>0$ with $\lim_{N\to\infty}\varepsilon_N=0$, and

$$\lim_{N\to\infty}\varepsilon_N\chi_N=\infty. \tag{5.30}$$

Next, let $\partial_N Q_t=\{x\in Q_t:\operatorname{dist}(x,\partial Q_t)\leqslant\varepsilon_N\}$ and $Q_{N,t}=Q_t\backslash\partial_N Q_t$. Moreover, we need for any $t\in[0,T]$ an approximation $q_N(\cdot,t)$ of $p(\cdot,t)$ satisfying

$$q_N(x,t)=p(x,t),\qquad x\in Q_{N,t}, \tag{5.31}$$

$$q_N(\cdot,t)\in C^3_b(\mathbb{R}), \tag{5.32}$$

$$\sup_{N\in\mathbb{N}}\sup_{t\leqslant T}(\|q_N(\cdot,t)\|_\infty+\|q'_N(\cdot,t)\|_\infty+\|q''_N(\cdot,t)\|_\infty)<\infty, \tag{5.33}$$

and

$$\lim_{N\to\infty} \sup_{t\leq T} \chi_N^{-1} \|q_N'''(\cdot, t)\|_\infty^2 = 0. \tag{5.34}$$

Note that such a function $q_N(\cdot, t)$ can easily be constructed by (P2), and since we can let ε_N tend to 0 arbitrarily slow.

We continue by deriving an analogue of (5.4). First, we have instead of (5.1)

$$\|h_{N,a}(\cdot, t)\|_2^2 = \|h_{N,a}(\cdot, 0)\|_2^2 - 2\int_0^t \langle X_{N,a}(s), g'_{N,a}(\cdot, s)\, g'_{N,a}(\cdot, s)\rangle\, ds$$
$$-\frac{1}{N} \sum_{\{k: t_N^k < t\}} \sum_{l\in L_N(t_N^k)} V_N(X_{N,a}^k(t_N^k) - X_{N,a}^l(t_N^k))(2-\delta_{k,l}), \tag{5.35}$$

where the final term counts the loss of $\|h_{N,a}(\cdot, t)\|_2^2 = \langle X_{N,a}(t), g_{N,a}(\cdot, t)\rangle$ due to absorption of the processes $X_{N,a}^k$ at the boundary ∂Q_t. Since by (2.10) this expression is negative, we obtain by (2.14), (2.22)

$$\sup_{N\in\mathbb{N}} \left(\sup_{t\geq 0} \|h_{N,a}(\cdot, t)\|_2^2 + \int_0^\infty \langle X_{N,a}(s), |g'_{N,a}(\cdot, s)|^2\rangle\, ds\right) < \infty. \tag{5.36}$$

Analogously to (5.35) we obtain

$$\langle h_{N,a}(\cdot, t), p(\cdot, t)\rangle$$
$$= \langle h_{N,a}(\cdot, 0), p_0\rangle - \int_0^t (\langle X_{N,a}(s), g'_{N,a}(\cdot, s)\, p'(\cdot, s) * W_N^-\rangle$$
$$+ \langle p(\cdot, s), p'(\cdot, s)\, h'_{N,a}(\cdot, s)\rangle)\, ds$$
$$-\frac{1}{N} \sum_{\{k: t_N^k < t\}} \int_{\mathbb{R}} p(x, t_N^k)\, W_N(x - X_{N,a}^k(t_N^k))\, dx. \tag{5.37}$$

In the same way as in the derivation of (5.4) we obtain from (5.3), (5.35), (5.37)

$$\|h_{N,a}(\cdot, t) - p(\cdot, t)\|_2^2$$
$$= \|h_{N,a}(\cdot, 0) - p_0\|_2^2 - 2\int_0^t \langle X_{N,a}(s), |g'_{N,a}(\cdot, s) - p'(\cdot, s)|^2\rangle\, ds$$
$$+ 2\int_0^t \langle X_{N,a}(s) - p(\cdot, s), p'(\cdot, s)(p'(\cdot, s) * W_N^- - g'_{N,a}(\cdot, s))\rangle\, ds$$

$$+2\int_0^t \{\langle X_{N,a}(s), (g'_{N,a}(\cdot,s)-p'(\cdot,s))(p'(\cdot,s) * W_N^- - p'(\cdot,s))\rangle$$

$$+\langle p(\cdot,s), p'(\cdot,s)(h'_{N,a}(\cdot,s)$$

$$- g'_{N,a}(\cdot,s) - p'(\cdot,s) + p'(\cdot,s) * W_N^-)\rangle\}\, ds$$

$$-\frac{1}{N}\sum_{\{k:t_N^k<t\}}\ \sum_{l\in L_N(t_N^k)} V_N(X_{N,a}^k(t_N^k) - X_{N,a}^l(t_N^k))(2-\delta_{k,l})$$

$$+\frac{2}{N}\sum_{\{k:t_N^k<t\}}\int_{\mathbb{R}} p(x,t_N^k)\, W_N(x - X_{N,a}^k(t_N^k))\, dx. \tag{5.38}$$

To a large extent the study of (5.38) is parallel to the investigation of (5.4) in Subsection 5.B. The main difference is that we now have to take into account boundary effects, since $p''(\cdot,t)$ and $p'''(\cdot,t)$ do not need to exist at the boundary ∂Q_t of $\operatorname{supp}(p(\cdot,t))$. These boundary effects are estimated in:

LEMMA (5.39).

(a) $$\lim_{N\to\infty}\sup_{s\leqslant T}\langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s}\rangle = 0, \tag{5.40}$$

(b) $$\lim_{N\to\infty}\sup_{s\leqslant T}\langle X_{N,a}(s), \mathbb{1}_{Q_{N,s}}|p'(\cdot,s) * W_N^- - p'(\cdot,s)|\rangle = 0. \tag{5.41}$$

Proof.

(a) $$\langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s}\rangle$$

$$\leqslant |\langle X_{N,a}(s), \mathbb{1}_{\underline{\partial}_N Q_s} * W_N^-\rangle|$$

$$+\int_{\mathbb{R}} X_{N,a}(s)(dx)\int_{\{y:|y-x|\geqslant \varepsilon_N\}} |W_N^-(x-y)|\, dy$$

$$(\underline{\partial}_N Q_s = \{x\in\mathbb{R}: \operatorname{dist}(x,\partial Q_s)\leqslant 2\varepsilon_N\})$$

$$\leqslant |\langle h_{N,a}(\cdot,s), \mathbb{1}_{\underline{\partial}_N Q_s}\rangle| + \int_{\{y:|y|\geqslant\varepsilon_N\}} \chi_N |W_1^-(\chi_N y)|\, dy$$

$$\leqslant \|h_{N,a}(\cdot,s)\|_2\, \|\mathbb{1}_{\underline{\partial}_N Q_s}\|_2 + \int_{\{z:|z|\geqslant \varepsilon_N\chi_N\}} |W_1^-(z)|\, dz.$$

Equation (5.40) follows by (2.4), (5.30), (5.36).

(b) $$\langle X_{N,a}(s), \mathbb{1}_{Q_{N,s}}|p'(\cdot,s) * W_N^- - p'(\cdot,s)|\rangle$$

$$\leqslant \langle X_{N,a}(s), \mathbb{1}_{Q_{N,s}}|q'_N(\cdot,s) * W_N^- - q'_N(\cdot,s)|\rangle$$

$$+\langle X_{N,a}(s), \mathbb{1}_{Q_{N,s}}|q'_N(\cdot,s) * W_N^- - p'(\cdot,s) * W_N^-|\rangle$$

(by (5.31))

$$\leqslant C\left(\chi_N^{-1} \sup_{N\in\mathbb{N}} \sup_{t\leqslant T} \|q''_N(\cdot,t)\|_\infty + \langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s}\rangle \right.$$
$$\left. + \int_{\mathbb{R}} X_{N,a}(s)(dx) \int_{\{y:|y-x|\geqslant \varepsilon_N\}} |W_N^-(x-y)|\,dy\right)$$
$$\text{(by (2.19), (5.6), (5.31), (5.33))}.$$

By (5.33) and the same arguments as in the proof of (5.40) we obtain (5.41).

Now, we have

$$\begin{aligned}
&|\langle p(\cdot,s), p'(\cdot,s)(h'_{N,a}(\cdot,s)-g'_{N,a}(\cdot,s)-p'(\cdot,s)+p'(\cdot,s)*W_N^-)\rangle|\\
&\quad\leqslant |\langle p'(\cdot,s)^2-p'(\cdot,s)^2*W_N, h_{N,a}(\cdot,s)-p(\cdot,s)\rangle|\\
&\qquad+|\langle p(\cdot,s)\,p''(\cdot,s)-(p(\cdot,s)\,p''(\cdot,s))*W_N, h_{N,a}(\cdot,s)-p(\cdot,s)\rangle|\\
&\quad\leqslant \|h_{N,a}(\cdot,s)-p(\cdot,s)\|_2^2+s_1(s,N)
\end{aligned}\tag{5.42}$$

with

$$\lim_{N\to\infty}\int_0^T s_1(s,N)\,ds=0 \tag{5.43}$$

(by (2.16), (2.19), (P2), and since $\lim_{N\to\infty}\|f-f*W_N\|_2=0$ for any $f\in L^2(\mathbb{R})$). Next, as in (5.9),

$$\begin{aligned}
&|\langle X_{N,a}(s), (g'_{N,a}(\cdot,s)-p'(\cdot,s))(p'(\cdot,s)*W_N^- - p'(\cdot,s))\rangle|\\
&\quad\leqslant \tfrac14\langle X_{N,a}(s), |g'_{N,a}(\cdot,s)-p'(\cdot,s)|^2\rangle\\
&\qquad+\langle X_{N,a}(s), |p'(\cdot,s)*W_N^- - p'(\cdot,s)|^2\rangle\\
&\quad\leqslant \tfrac14\langle X_{N,a}(s), |g'_{N,a}(\cdot,s)-p'(\cdot,s)|^2\rangle + C(\langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s}\rangle\\
&\qquad+\langle X_{N,a}(s), \mathbb{1}_{Q_{N,s}}|p'(\cdot,s)*W_N^- - p'(\cdot,s)|\rangle)\\
&\qquad\text{(by (2.19))}.
\end{aligned}\tag{5.44}$$

For the integrand of the third term on the right side of (5.38) we obtain

$$\begin{aligned}
&|\langle X_{N,a}(s)-p(\cdot,s), p'(\cdot,s)(p'(\cdot,s)*W_N^- - g'_{N,a}(\cdot,s))\rangle|\\
&\quad\leqslant |\langle X_{N,a}(s)-p(\cdot,s), q'_N(\cdot,s)(p'(\cdot,s)*W_N^- - g'_{N,a}(\cdot,s))\rangle|\\
&\qquad+|\langle X_{N,a}(s), (p'(\cdot,s)-q'_N(\cdot,s))(p'(\cdot,s)*W_N^- - g'_{N,a}(\cdot,s))\rangle|\\
&\qquad+|\langle p(\cdot,s), (p'(\cdot,s)-q'_N(\cdot,s))(p'(\cdot,s)*W_N^- - g'_{N,a}(\cdot,s))\rangle|.
\end{aligned}\tag{5.45}$$

For the expressions on the right side of this inequality we get

$$
\begin{aligned}
&|\langle p(\cdot, s), (p'(\cdot, s) - q_N'(\cdot, s))(p'(\cdot, s) * W_N^- - g_{N,a}'(\cdot, s))\rangle| \\
&\quad \leqslant |\langle p'(\cdot, s), (p'(\cdot, s) - q_N'(\cdot, s))(p(\cdot, s) * W_N^- - g_{N,a}(\cdot, s))\rangle| \\
&\qquad + |\langle p(\cdot, s), (p''(\cdot, s) - q_N''(\cdot, s))(p(\cdot, s) * W_N^- - g_{N,a}(\cdot, s))\rangle| \\
&\quad \leqslant C\langle \mathbb{1}_{\partial_N Q_s}, |p(\cdot, s) * W_N^- - g_{N,a}(\cdot, s)|\rangle \\
&\qquad\qquad \text{(by (2.16), (2.19), (P2), (5.31), (5.33))} \\
&\quad \leqslant C(\|\mathbb{1}_{\partial_N Q_s}\|_2^2 + \|p(\cdot, s) * W_N^- - g_{N,a}(\cdot, s)\|_2^2) \\
&\quad \leqslant C(\|h_{N,a}(\cdot, s) - p(\cdot, s)\|_2^2 + \varepsilon_N),
\end{aligned} \tag{5.46}
$$

$$
\begin{aligned}
&|\langle X_{N,a}(s), (p'(\cdot, s) - q_N'(\cdot, s))(p'(\cdot, s) * W_N^- - g_{N,a}'(\cdot, s))\rangle| \\
&\quad \leqslant C\langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s}\rangle^{1/2} \\
&\qquad \times \langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s} |p'(\cdot, s) * W_N^- - g_{N,a}'(\cdot, s)|^2\rangle^{1/2} \\
&\qquad\qquad \text{(by (2.19), (5.31), (5.33))} \\
&\quad \leqslant \tfrac{1}{4}\langle X_{N,a}(s), |p'(\cdot, s) - g_{N,a}'(\cdot, s)|^2\rangle + C\langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s}\rangle,
\end{aligned} \tag{5.47}
$$

and

$$
\begin{aligned}
&|\langle X_{N,a}(s) - p(\cdot, s), q_N'(\cdot, s)(p'(\cdot, s) * W_N^- - g_{N,a}'(\cdot, s))\rangle| \\
&\quad \leqslant C((\|h_{N,a}(\cdot, s) - p(\cdot, s)\|_2^2 + \chi_N^{-2})(\|q_N''(\cdot, s)\|_\infty + 1) \\
&\qquad + \chi_N^{-1}\, \|q_N'''(\cdot, s)\|_\infty^2) \\
&\qquad \text{(in exactly the same way as in the proof of (5.26))} \\
&\quad \leqslant C(\|h_{N,a}(\cdot, s) - p(\cdot, s)\|_2^2 + \chi_N^{-2} + \chi_N^{-1}\, \|q_N'''(\cdot, s)\|_\infty^2) \\
&\qquad\qquad \text{(by (5.33)).}
\end{aligned} \tag{5.48}
$$

Finally, (2.17) implies $p(X_{N,a}^k(t_N^k), t_N^k) = 0$, and therefore

$$
\begin{aligned}
&\frac{1}{N} \sum_{\{k : t_N^k < t\}} \int_{\mathbb{R}} p(x, t_N^k)\, W_N(x - X_{N,a}^k(t_N^k))\, dx \\
&\quad \leqslant \frac{C}{N} \sum_{\{k : t_N^k < t\}} \int_{\mathbb{R}} |x - X_{N,a}^k(t)|\, W_N(x - X_{N,a}^k(t))\, dx \leqslant \frac{C}{\chi_N} \\
&\qquad\qquad \text{(by (2.4), (2.19)).}
\end{aligned} \tag{5.49}
$$

Since the fifth expression on the right side of (5.38) is negative, we obtain from (5.34), (5.38)–(5.49)

$$\|h_{N,a}(\cdot, t) - p(\cdot, t)\|_2^2$$
$$\leqslant \|h_{N,a}(\cdot, 0) - p_0\|_2^2 - \int_0^t \langle X_{N,a}(s), |g'_{N,a}(\cdot, s) - p'(\cdot, s)|^2\rangle \, ds$$
$$+ C \int_0^t \|h_{N,a}(\cdot, s) - p(\cdot, s)\|_2^2 \, ds + s(t, N), \tag{5.50}$$

where

$$\lim_{N \to \infty} \sup_{t \leqslant T} s(t, N) = 0. \tag{5.51}$$

The proof of Proposition (5.28) is finished by an application of Gronwall's Lemma.

Remark (5.52). In the system X_N without absorption at any instant $t \geqslant 0$ the processes $X_N^k(t)$, which have crossed the boundary at this moment, exert a repelling force on those processes, which remained inside Q_t, and therefore diminish the flux across ∂Q_t. On the other hand, in the system $X_{N,a}$ this repelling force is absent, since the processes, which leave $\operatorname{int}(Q_t)$, disappear. Therefore, we can expect that in $X_{N,a}$ more particles can reach $\mathbb{R} \backslash Q_t$.

However, since $\langle p(\cdot, t), 1\rangle \equiv 1$, (5.29) implies that the number of those processes $X_{N,a}^k$ in the system $X_{N,a}$, which actually disappear in the time interval $[0, T]$, is of order $o(N)$ as $N \to \infty$. Hence, we formally obtain

$$\lim_{N \to \infty} \sup_{t \leqslant T} \langle X_N(t), \mathbb{1}_{\mathbb{R} \backslash Q_t}\rangle$$
$$\lesssim \lim_{N \to \infty} \sup_{t \leqslant T} (1 - \langle X_{N,a}(t), 1\rangle) = 0. \tag{5.53}$$

A careful check of the proof of Proposition (5.28) reveals that the absorption of the individual processes in $\mathbb{R} \backslash \operatorname{int}(Q_s)$ was needed mathematically in effect only to have $\langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s}\rangle$ instead of $\langle X_{N,a}(s), \mathbb{1}_{\mathbb{R} \backslash Q_{N,s}}\rangle$ in inequality (5.47). If we could replace a priori the estimate (5.40) by (5.53), or more precisely by

$$\lim_{N \to \infty} \sup_{t \leqslant T} \langle X_N(t), \mathbb{1}_{\mathbb{R} \backslash Q_{N,t}}\rangle = 0, \tag{5.54}$$

we could drop this rather artificial modification of the dynamics of the processes X_N^k.

In particular, this means that we could replace the estimate (5.47) by an analogous estimate involving $\langle X_N(s), \mathbb{1}_{\mathbb{R}\setminus Q_{N,s}}\rangle$ instead of $\langle X_{N,a}(s), \mathbb{1}_{\partial_N Q_s}\rangle$. Finally, we would obtain (5.50), (5.51) with X_N, h_N, g_N replacing $X_{N,a}$, $h_{N,a}$, $g_{N,a}$.

This observation will be a key ingredient in the completion of the proof of Theorem (2.21).

E. *Completion of the Proof of Theorem* (2.21)

As indicated in Remark (5.52), it suffices to know a priori that $\langle X_N(t), \mathbb{1}_{\mathbb{R}\setminus Q_{N,t}}\rangle$ tends to 0 as $N \to \infty$, and then apply the technique of the proof of Proposition (5.28).

Let us use the notation $Q_t = [b_-(t), b_+(t)]$, and then introduce another two sequences of auxiliary systems $X^{r;}_{N,a}$, $X^{l;}_{N,a}$, $N \in \mathbb{N}$, of interacting particles. We define

$$X^{r;}_{N,a}(t) = \frac{1}{N} \sum_{k \in L^r_N(t)} \delta_{X^{r;k}_{N,a}(t)},$$

where $X^{r;k}_{N,a}(\cdot) = X^{r;k}_{N,a}(t)$, $0 \leqslant t \leqslant t^{r;k}_N$, solves

$$\frac{d}{dt} X^{r;k}_{N,a}(t) = -\frac{1}{N} \sum_{\substack{m \in L^r_N(t)\\ m \neq k}} V'_N(X^{r;k}_{N,a}(t) - X^{r;m}_{N,a}(t)),$$

$$X^{r;k}_{N,a}(0) = X^k_N(0),$$

with

$$t^{r;k}_N = \inf\{t \geqslant 0 : X^{r;k}_{N,a}(t) \geqslant b_+(t)\},$$

$$L^r_N(t) = \{k \in \{1, ..., N\} : t^{r;k}_N \geqslant t\}.$$

Hence, $X^{r;}_{N,a}$ is obtained from X_N by deleting those processes X^k_N which have reached $[b_+(\cdot), \infty)$.

In quite the same way we define $X^{l;}_{N,a}$, $X^{l;k}_{N,a}$, etc., except that we replace b_+ by b_-, i.e., $X^{l;}_{N,a}$ is obtained from X_N by absorbing those particles, which have reached $(-\infty, b_-(\cdot)]$.

To finish the proof of Theorem (2.21) we need four steps:

(i) First, we derive

$$\langle X^{r;}_{N,a}(t), \mathbb{1}_{(-\infty, b_-(t))}\rangle \leqslant 1 - \langle X_{N,a}(t), 1\rangle, \qquad t \geqslant 0. \tag{5.55}$$

To prove (5.55) we contend that for any $t \geqslant 0$, $m = 1, ..., N$, one of the following alternatives holds.

$$\begin{aligned} &(1) \quad t_N^{r;m},\ t_N^m > t \text{ and } X_{N,a}^{r;m}(t) \geqslant X_{N,a}^m(t), \\ &(2) \quad X_{N,a}^{r;m}(s) \text{ has been absorbed in } [b_+(s), \infty) \text{ for some } s \leqslant t, \\ &(3) \quad X_{N,a}^m(s) \text{ has been absorbed in } (-\infty, b_-(s)] \text{ for some } s \leqslant t. \end{aligned} \tag{5.56}$$

To prove this we define $t^* = \inf\{t \geqslant 0 : (5.56) \text{ does not hold}\}$. Then, we replace for a few moments $V_1(x)$ by $V_1(x) + \varepsilon \exp(-|x|)$, $\varepsilon > 0$, such that instead of (2.8) we get

$$V_1''(x) > 0, \qquad x \neq 0. \tag{5.57}$$

Obviously, the resulting empirical processes $X_{N,a}$ and $X_{N,a}^{r;}$ satisfy

$$\begin{aligned} &X_{N,a}(t) = X_{N,a}^{r;}(t), \\ &0 \leqslant t \leqslant \tau_{N,1} = \inf\{t \geqslant 0 : X_{N,a}^k(t) \leqslant b_-(t) \text{ for some } k = 1, ..., N\}. \end{aligned}$$

Inequality (5.57) implies

$$V_1'(x)\,\mathrm{sgn}(-x) > 0, \qquad x \neq 0, \tag{5.58}$$

and therefore the loss of a particle in $X_{N,a}$ at time $\tau_{N,1}$ yields

$$\begin{aligned} &\lim_{s \searrow \tau_{N,1}} -\frac{1}{N} \sum_{\substack{m \in L_N(s) \\ m \neq k}} V_N'(X_{N,a}^k(s) - X_{N,a}^m(s)) \\ &\quad < \lim_{s \searrow \tau_{N,1}} -\frac{1}{N} \sum_{\substack{m \in L_N^r(s) \\ m \neq k}} V_N'(X_{N,a}^{r;k}(s) - X_{N,a}^{r;m}(s)), \qquad k \in \bigcup_{s \searrow \tau_{N,1}} L_N(s). \end{aligned}$$

Consequently,

$$\lim_{s \searrow \tau_{N,1}} \frac{d}{ds} (X_{N,a}^k(s) - X_{N,a}^{r;k}(s)) < 0, \qquad k \in \bigcup_{s \searrow \tau_{N,1}} L_N(s),$$

i.e.,

$$\tau_{N,1} < t^*. \tag{5.59}$$

Furthermore,

$$\begin{aligned} &\langle X_{N,a}(t), \mathbb{1}_{(-\infty, X_{N,a}^k(t)]} \rangle < \langle X_{N,a}^{r;}(t), \mathbb{1}_{(-\infty, X_{N,a}^{r;k}(t)]} \rangle, \\ &\qquad\qquad k \in L_N(t) \cap L_N^r(t), \quad \tau_{N,1} < t; \end{aligned} \tag{5.60}$$

i.e., on the left of any $X_{N,a}^k(t)$ the system $X_{N,a}(t)$ has strictly less particles than the system $X_{N,a}^{r;}(t)$ on the left of the corresponding particle $X_{N,a}^{r;k}(t)$. Note that (5.56) can start to be violated at t^* if and only if the first alternative stops to hold at this instant. Since the trajectories of the processes

$t \to X^{r;m}_{N,a}(t)$, and $t \to X^m_{N,a}(t)$, respectively, are left-continuous, (5.56) still holds for t^*. Moreover, there exists at least one k^*, such that $t^{r;k^*}_N$, $t^{k^*}_N > t^*$, and $X^{r;k^*}_{N,a}(t^*) = X^{k^*}_{N,a}(t^*)$. By (5.56)–(5.58) we conclude

$$-V'_N(X^{r;k^*}_{N,a}(t^*) - X^{r;m}_{N,a}(t^*))\, \mathbb{1}_{[t^*,\infty)}(t^{r;m}_N)$$
$$\geqslant -V'_N(X^{k^*}_{N,a}(t^*) - X^m_{N,a}(t^*))\, \mathbb{1}_{[t^*,\infty)}(t^m_N), \qquad m \in \{1, ..., N\} \backslash \{k^*\}.$$

Summing over m and taking into account (5.58), (5.60) we obtain

$$-\frac{1}{N} \sum_{\substack{m \in L^r_N(t^*) \\ m \neq k^*}} V'_N(X^{r;k^*}_{N,a}(t^*) - X^{r;m}_{N,a}(t^*))$$
$$> -\frac{1}{N} \sum_{\substack{m \in L_N(t^*) \\ m \neq k^*}} V'_N(X^{k^*}_{N,a}(t^*) - X^m_{N,a}(t^*)),$$

or, in other words,

$$\lim_{s \to t^*} \frac{d}{ds} (X^{k^*}_{N,a}(s) - X^{r;k^*}_{N,a}(s)) < 0.$$

These considerations show $t^* = \infty$; i.e., (5.56) holds for any $t \geqslant 0$. Of course, this conclusion remains true in the limit $\varepsilon \to 0$. Inequality (5.55) is an obvious consequence of (5.56).

(ii) In a second step we show

$$\lim_{N \to \infty} \left[\sup_{t \leqslant T} \|h^r_{N,a}(\cdot, t) - p(\cdot, t)\|_2^2 + \int_0^T \langle X^r_{N,a}(s), |(g^r_{N,a})'(\cdot, s) - p'(\cdot, s)|^2 \rangle \, ds \right] = 0, \tag{5.61}$$

where $h^r_{N,a}(\cdot, t) = (X^r_{N,a}(t) * W_N)(\cdot)$ and $g^r_{N,a}(\cdot, t) = (X^r_{N,a}(t) * V_N)(\cdot)$. By Proposition (5.28) and Subsection 5.C we obtain the validity of (2.24) for $X_{N,a}$. Hence,

$$\lim_{N \to \infty} \sup_{t \leqslant T} (1 - \langle X_{N,a}(t), 1 \rangle) = \lim_{N \to \infty} \sup_{t \leqslant T} \langle p(\cdot, t) - X_{N,a}(t), 1 \rangle = 0,$$

i.e., by (5.55)

$$\lim_{N \to \infty} \sup_{t \leqslant T} \langle X^r_{N,a}(t), \mathbb{1}_{(-\infty, b_-(t))} \rangle = 0. \tag{5.62}$$

To prove (5.61) we take advantage of the observation noted in Remark (5.52). This means, we replace the estimate (5.40) by (5.62) and

$$\lim_{N \to \infty} \sup_{t \leqslant T} [\langle X^r_{N,a}(t), \mathbb{1}_{[b_+(t) - \varepsilon_N, b_+(t)]} \rangle + \langle X^r_{N,a}(t), \mathbb{1}_{[b_-(t), b_-(t) + \varepsilon_N]} \rangle] = 0, \tag{5.63}$$

which can be proved in the same manner as (5.40). Then we can repeat the proof of Proposition (5.28) word by word to obtain finally (5.61).

(iii) Quite analogously to (5.55) we derive

$$\langle X_N(t), \mathbb{1}_{(b_+(t),\infty)} \rangle \leqslant 1 - \langle X^{r_i}_{N,a}(t), 1 \rangle, \qquad t \geqslant 0. \tag{5.64}$$

By arguments similar to those leading to (5.62) we can conclude from (5.61), (5.64)

$$\lim_{N \to \infty} \sup_{t \leqslant T} \langle X_N(t), \mathbb{1}_{(b_+(t),\infty)} \rangle = 0. \tag{5.65}$$

By employing the auxiliary system $X^{l_i}_{N,a}$ we then derive in the same way

$$\lim_{N \to \infty} \sup_{t \leqslant T} \langle X_N(t), \mathbb{1}_{(-\infty,b_-(t))} \rangle = 0. \tag{5.66}$$

(iv) Estimates (5.65), (5.66), and (5.63) with X_N replacing $X^{r_i}_{N,a}$ yield (5.54). As noted in Remark (5.52), we are able now to derive (5.50), (5.51), and hence to finish the proof of Theorem (2.21).

6. Proof of Theorem (3.8)

A. *Regularity of $X_N(t)$ for $t > 0$*

It will be very helpful that the regularity properties (3.3)–(3.6) essentially hold for positive times too.

Lemma (6.1). *Let $d^k_N(t) = X^k_N(t) - X^{k-1}_N(t)$, $k = 2, ..., N$. Then*

(i) *$X_N(t)$ is symmetric with respect to 0,*

$$\textit{and } X^1_N(t) \leqslant X^2_N(t) \leqslant \cdots \leqslant X^N_N(t) \textit{ for any } t \geqslant 0, \tag{6.2}$$

(ii) *$d^k_N(t)$ is decreasing in $k \in \{2, ..., [(N+2)/2]\}$ for any $t \geqslant 0$,* (6.3)

(iii) $\dfrac{d}{dt} X^k_N(t) \leqslant 0,\ k = 1, ..., [(N+1)/2],\ t \geqslant 0,$ (6.4)

(iv) $d_0 = \inf\{N d^k_N(t) : t \geqslant 0;\ N \in \mathbb{N};\ k = 2, ..., N\} > 0,$ (6.5)

(v) *for $c > 0$ let $K_N(c) = \{k \in \{2, ..., N\} : \sup_{t \leqslant T} N d^k_N(t) \geqslant c\}$, then*

$$\sup_{N \in \mathbb{N}, c \geqslant 0} \frac{|K_N(c)|\, c^{2/3}}{N} < \infty. \tag{6.6}$$

$$\text{(vi)}\quad \sup_{N\in\mathbb{N}}\left(\sup_{t\geqslant 0}\frac{1}{N^2}\sum_{\substack{k,m=1\\k\neq m}}^{N} V_N(X_N^k(t)-X_N^m(t))\right.$$
$$\left.+\int_0^\infty \frac{1}{N}\sum_{k=1}^{N}\left|\frac{1}{N}\sum_{\substack{m=1\\m\neq k}}^{N} V'_N(X_N^k(s)-X_N^m(s))\right|^2 ds\right)<\infty. \tag{6.7}$$

In (6.6) we denote by $|A|$ the cardinality of a finite set A. In other contexts we use the same notation $|B|$ for the Lebesgue measure of some set $B\subseteq\mathbb{R}$, or $|x|$ for the absolute value of $x\in\mathbb{R}$. Confusion about this ambiguous use of $|\cdot|$ should not arise.

Remark. Equation (6.5) states that the particles cannot come arbitrarily close to each other. Therefore, without restricting generality we can and do replace (3.1), (3.2) immediately after the proof of (6.5) by

$$\begin{aligned} &V_1\in C_b^4(\mathbb{R}); \qquad V_1^{(4)}(x)>0, \qquad |x|>d_0/2,\\ &|V_1^{(k)}(x)|\leqslant C(1+|x|)^{-2-k}, \qquad k=0,1,2,\quad x\in\mathbb{R}. \end{aligned} \tag{6.8}$$

Moreover, we then can use the representation (1.7) of (1.1) throughout the remainder of this section.

Proof. First, let us note that (3.1), (3.2) imply

$$V_1^{(k)}(x)\,\mathrm{sgn}(-x)^k>0, \qquad k=0,...,4,\quad x\in\mathbb{R}\setminus\{0\}. \tag{6.9}$$

In particular, (2.8) is satisfied, and $\mathrm{supp}(V_1)=\mathbb{R}$.

(i) This is an obvious consequence of (3.3), the symmetry of V_1, and the fact that trajectories of different particles cannot cross.

(ii) Let $t^*=\inf\{t\geqslant 0: d_N^k(t)>d_N^{k-1}(t)$ for some $k=3,...,[(N+2)/2]\}$. The continuity of the processes $t\to X_N^k(t)$ and (3.4) imply

$$d_N^k(t^*)\leqslant d_N^{k-1}(t^*),\ k=3,...,[(N+2)/2], \tag{6.10}$$

with equality for at least one k, say $k^*\in\{3,...,[(N+2)/2]\}$. Therefore it suffices to prove

$$\frac{d}{dt}(d_N^{k^*}(t^*)-d_N^{k^*-1}(t^*))<0. \tag{6.11}$$

We obtain from (1.1)

$$N\frac{d}{dt}(d_N^{k^*}(t^*)-d_N^{k^*-1}(t^*))$$
$$=N\frac{d}{dt}(X_N^{k^*}(t^*)-2X_N^{k^*-1}(t^*)+X_N^{k^*-2}(t^*))$$

$$
\begin{aligned}
&= -\sum_{l \geqslant k^*+1} V'_N(X_N^{k^*}(t^*) - X_N^l(t^*)) - V'_N(d_N^{k^*}(t^*)) \\
&\quad - \sum_{l \leqslant k^*-2} V'_N(X_N^{k^*}(t^*) - X_N^l(t^*)) \\
&\quad -2\Big[-V'_N(-d_N^{k^*}(t^*)) - \sum_{l \geqslant k^*+1} V'_N(X_N^{k^*-1}(t^*) - X_N^l(t^*)) \\
&\qquad - V'_N(d_N^{k^*-1}(t^*)) - \sum_{l \leqslant k^*-3} V'_N(X_N^{k^*-1}(t^*) - X_N^l(t^*))\Big] \\
&\quad - V'_N(-d_N^{k^*-1}(t^*)) - \sum_{l \geqslant k^*} V'_N(X_N^{k^*-2}(t^*) - X_N^l(t^*)) \\
&\quad - \sum_{l \leqslant k^*-3} V'_N(X_N^{k^*-2}(t^*) - X_N^l(t^*)) \\
&= -\sum_{l \geqslant k^*+1} [V'_N(X_N^{k^*}(t^*) - X_N^l(t^*)) - 2V'_N(X_N^{k^*-1}(t^*) - X_N^l(t^*)) \\
&\qquad + V'_N(X_N^{k^*-2}(t^*) - X_N^l(t^*))] \\
&\quad - \sum_{l \leqslant k^*-3} [V'_N(X_N^{k^*}(t^*) - X_N^l(t^*)) - 2V'_N(X_N^{k^*-1}(t^*) - X_N^l(t^*)) \\
&\qquad + V'_N(X_N^{k^*-2}(t^*) - X_N^l(t^*))] \\
&\quad - V'_N(X_N^{k^*-2}(t^*) - X_N^{k^*}(t^*)) - V'_N(X_N^{k^*}(t^*) - X_N^{k^*-2}(t^*)) \\
&\qquad (\text{since } d_N^{k^*}(t^*) = d_N^{k^*-1}(t^*)) \\
&= -\sum_{n=2}^{k^*-2} \{[V'_N(X_N^{k^*}(t^*) - X_N^{k^*-1+n}(t^*)) \\
&\qquad - 2V'_N(X_N^{k^*-1}(t^*) - X_N^{k^*-1+n}(t^*)) \\
&\qquad + V'_N(X_N^{k^*-2}(t^*) - X_N^{k^*-1+n}(t^*))] \\
&\qquad + [V'_N(X_N^{k^*}(t^*) - X_N^{k^*-1+n}(t^*)) \\
&\qquad - 2V'_N(X_N^{k^*-1}(t^*) - X_N^{k^*-1-n}(t^*)) \\
&\qquad + V'_N(X_N^{k^*-2}(t^*) - X_N^{k^*-1-n}(t^*))]\} \\
&\quad - \sum_{l \geqslant 2k^*-2} [V'_N(X_N^{k^*}(t^*) - X_N^l(t^*)) - 2V'_N(X_N^{k^*-1}(t^*) - X_N^l(t^*)) \\
&\qquad + V'_N(X_N^{k^*-2}(t^*) - X_N^l(t^*))]. \qquad (6.12)
\end{aligned}
$$

For fixed h, x, y with $|x-y| > h > 0$ we have

$$
\begin{aligned}
&V'_N(x+h-y) - 2V'_N(x-y) + V'_N(x-h-y) \\
&\quad = \int_x^{x+h} du \int_{-h}^0 dv\, V'''_N(u+v-y). \qquad (6.13)
\end{aligned}
$$

By (6.9) this expression is strictly positive for $y > x+h$ implying that the second sum in (6.12) is strictly negative.

Next, we conclude from (6.9), (6.13), and the symmetry of V_N that the function $y \to V'_N(x+h-y) - 2V'_N(x-y) + V'_N(x-h-y)$ is antisymmetric with respect to x and satisfies for $|x-y| > h$

$$\frac{d}{dy}\left[V'_N(x+h-y) - 2V'_N(x-y) + V'_N(x-h-y)\right] = -\int_x^{x+h} du \int_{-h}^{0} dv\, V_N^{(4)}(u+v-y) < 0.$$

Therefore, since by (6.2), (6.10) for any $n = 2, ..., k^* - 2$

$$0 \leqslant X_N^{k^*-1+n}(t^*) - X_N^{k^*-1}(t^*) \leqslant X_N^{k^*-1}(t^*) - X_N^{k^*-1-n}(t^*),$$

the first sum in (6.12) is also negative.
Hence, (6.11) and therefore (6.3) too, follows.

(iii) Equation (1.1) implies for $k = 1, ..., [(N+1)/2]$

$$\frac{d}{dt} X_N^k(t) = -\sum_{n=1}^{k-1} \left[V'_N(X_N^k(t) - X_N^{k+n}(t)) + V'_N(X_N^k(t) - X_N^{k-n}(t))\right] - \sum_{l=2k}^{N} V'_N(X_N^k(t) - X_N^l(t)).$$

Both sums in this expression are negative by (6.9) and

$$0 \leqslant X_N^{k+n}(t) - X_N^k(t) \leqslant X_N^k(t) - X_N^{k-n}(t), \qquad n = 1, ..., k-1, \quad t \geqslant 0,$$

which follows from (6.2), (6.3).

(iv) Relations (6.2), (6.3) imply for any $t \geqslant 0$ and $N \in \mathbb{N}$

$$\inf\{d_N^k(t) : k = 2, ..., N\} = \begin{cases} d_N^{N/2+1}(t), & \text{if } N \text{ is even}, \\ d_N^{(N+1)/2}(t), & \text{if } N \text{ is odd}. \end{cases}$$

For N even we have

$$\frac{d}{dt} d_N^{N/2+1}(t) = \frac{d}{dt} X_N^{N/2+1}(t) - \frac{d}{dt} X_N^{N/2}(t) = -2\frac{d}{dt} X_N^{N/2}(t) \geqslant 0 \qquad \text{(by (6.2), (6.4))}.$$

Hence (6.5) for N even follows from (3.5).
The case N odd is handled in the same way.

(vi) First, (6.5), (6.8) imply

$$g_N(x,t) \leqslant \frac{1}{N} \sum_{k=1}^{N} CN(1+N\,|x-X_N^k(t)|)^{-2}$$

$$\leqslant C \sum_{k=-N}^{N} (1+d_0\,|k|)^{-2} < \infty, \qquad \text{uniformly in } x \in \mathbb{R},\ t \geqslant 0,\ N \in \mathbb{N}. \tag{6.14}$$

Hence, (6.7) follows from

$$\frac{1}{N^2} \sum_{k,l=1}^{N} V_N(X_N^k(t) - X_N^l(t)) = \langle X_N(t), g_N(\cdot, t)\rangle$$

$$= \frac{1}{N^2} \sum_{k,l=1}^{N} V_N(X_N^k(0) - X_N^l(0)) - 2\int_0^t \langle X_N(s), |g_N'(\cdot, s)|^2\rangle\, ds,$$

which is a consequence of (1.7) (cf. (5.1)).

(v) Condition (3.6) implies that

$$\operatorname{supp}(X_N(0)) \subseteq [-K, K] \qquad \text{for some} \quad K > 0 \text{ uniformly in } N. \tag{6.15}$$

We obtain for $m \geqslant 2$

$$k_N^*(mK) = \sup\left\{k \in \left\{1, ..., \left[\frac{N+1}{2}\right]\right\} : \sup_{t \leqslant T} |X_N^k(t)| \geqslant mK\right\}$$

$$= \frac{1}{2}\,|\{k \in \{1, ..., N\} : \sup_{t \leqslant T} |X_N^k(t)| \geqslant mk\}| \qquad \text{(by (6.2))}$$

$$\leqslant \left|\left\{k : \int_0^T \left|\frac{1}{N} \sum_{\substack{m=1\\ m\neq k}}^{N} V_N'(X_N^k(s) - X_N^m(s))\right| ds > (m-1)\,K\right\}\right|$$

$$\text{(by (1.1), (6.15))}$$

$$\leqslant \left|\left\{k : \int_0^T \left|\frac{1}{N} \sum_{\substack{m=1\\ m\neq k}}^{N} V_N'(X_N^k(s) - X_N^m(s))\right|^2 ds > (m-1)^2 K^2/T\right\}\right|$$

$$\leqslant N \frac{T}{(m-1)^2 K^2} \frac{1}{N} \sum_{k=1}^{N} \int_0^T \left|\frac{1}{N} \sum_{\substack{m=1\\ m\neq k}}^{N} V_N'(X_N^k(s) - X_N^m(s))\right|^2 ds$$

$$\leqslant C \frac{N}{m^2 K^2} \qquad \text{(by (6.7))}. \tag{6.16}$$

Let $K_N^*(mk) = \{k_N^*(mK) + 2, ..., N - k_N^*(mk) - 1\}$ and $K_N(c, t) = \{k \in \{2, ..., N\}: Nd_N^k(t) \geq c\}$, $t \geq 0$. Obviously, $|K_N(c, t) \cap K_N^*(mK)| \, c/N \leq 2mK$. Hence, by (6.16)

$$|K_N(c, t)| \leq |\{1, ..., N\} \backslash K_N^*(mK)| + |K_N^*(mK) \cap K_N(c, t)|$$
$$\leq C\left(\frac{N}{(mK)^2} + \frac{NmK}{c}\right) \leq CNc^{-2/3}, \text{ uniformly in } t \in [0, T],$$

if $c \geq 8$, and if we choose $m = c^{1/3}$. For $c \in (0, 8]$ we obviously have $|K_N(c, t)| \leq 4Nc^{-2/3}$.

Due to (6.2), (6.3), $K_N(c, t)$ is for any $t \geq 0$ the complement of an "interval" in $\{2, ..., N\}$, which is symmetric with respect to $(N+2)/2$. Therefore, there exists some $\underline{t}_N$ with $K_N(c) = K_N(c, \underline{t}_N)$. Hence, the proof of (6.6), and therefore the proof of Lemma (6.1) too, is finished.

B. *The Local Equilibrium*

We show that *for most of the time most of the particles are locally almost equidistant.* For that we utilize the fact that the inter-particle forces are strongly repulsive, at least, if the distance between neighbouring particles is not too large.

LEMMA (6.17). (i) *For* $c > 0$, $t \geq 0$ *let* $k_N(c, t) = \inf\{k \in \{2, ..., N\}: Nd_N^k(t) < c\}$ *and* $k_N(c) = \sup_{t \leq T} k_N(c, t) = \inf\{k \in \{2, ..., N\} \backslash K_N(c)\}$. *Then*

$$\sup_{N \in \mathbb{N}} \int_0^\infty \frac{1}{N} \sum_{k = k_N(c, t)}^{N - k_N(c, t) + 1} N^4 \, |d_N^{k+1}(t) - d_N^k(t)|^2 \, dt \leq CV_1''(c)^{-2}. \quad (6.18)$$

(ii) *For* $\alpha, c > 0$ *let*

$$T_N(\alpha, c) = \left\{t \leq T : d_N^{k+1}(t) \leq d_N^k(t) - \alpha \text{ for some } k \in \left\{k_N(c), ..., \left[\frac{N+1}{2}\right]\right\}\right\}.$$

We set $\varepsilon = 0.001$, *and define* $\alpha_N = N^{\varepsilon - 3/2}$, *and* $\underline{c}_N \to \infty$, *such that*

$$\max\{\underline{c}_N, V_1''(\underline{c}_N)^{-2}\} = N^\varepsilon. \quad (6.19)$$

Then

$$\lim_{N \to \infty} |T_N(\alpha_N, \underline{c}_N)| = 0, \quad (6.20)$$

and

$$\begin{aligned} &|X_N^k(t) - X_N^l(t) - (k-l) \, d_N^l(t)| \leq C(k-l)^2 \alpha_N, \\ &k, l \in \{2, ..., N\} \backslash K_N(\underline{c}_N), \qquad t \in [0, T] \backslash T_N(\alpha_N, \underline{c}_N). \end{aligned} \quad (6.21)$$

Note that by (6.9) the estimate (6.18) is nontrivial for any $c>0$. Moreover, (6.9) is needed for the construction of a sequence $\underline{c}_N \to \infty$ satisfying (6.19).

Proof. (i) For any $t \geqslant 0$

$$\begin{aligned}\langle X_N(t), |g'_N(\cdot, t)|^2\rangle &= \frac{1}{N}\sum_{k=1}^{N}\left|\frac{1}{N}\sum_{\substack{l=1\\ l\neq k}}^{N} V'_N(X_N^k(t)-X_N^l(t))\right|^2 \\ &\geqslant \frac{1}{N}\sum_{k=k_N(c,t)}^{[N/2]}\left|\frac{1}{N}\sum_{n=1}^{k-1}[V'_N(X_N^k(t)-X_N^{k-n}(t)) \right.\\ &\qquad + V'_N(X_N^k(t)-X_N^{k+n}(t))] \\ &\qquad \left. +\frac{1}{N}\sum_{l=2k}^{N} V'_N(X_N^k(t)-X_N^l(t))\right|^2.\end{aligned}$$

By (6.2), (6.3), (6.9) each single term in the first and second inner sum in the last expression is positive. Hence, we obtain uniformly in $N \in \mathbb{N}$

$$\begin{aligned}\infty &> \int_0^\infty \langle X_N(t), |g'_N(\cdot, t)|^2\rangle\, dt \qquad \text{(by (6.7))} \\ &\geqslant \int_0^\infty \frac{1}{N^3}\sum_{k=k_N(c,t)}^{[N/2]} |V'_N(X_N^k(t)-X_N^{k-1}(t)) + V'_N(X_N^k(t)-X_N^{k+1}(t))|^2\, dt \\ &= \int_0^\infty \frac{1}{N^3}\sum_{k=k_N(c,t)}^{[N/2]} N^4\, |V'_1(Nd_N^k(t)) - V'_1(Nd_N^{k+1}(t))|^2\, dt \qquad (6.22) \\ &\qquad \text{(by the symmetry of } V_1) \\ &\geqslant \int_0^\infty \frac{1}{N}\sum_{k=k_N(c,t)}^{[N/2]} N^4 V''_1(c)^2\, |d_N^k(t) - d_N^{k+1}(t)|^2\, dt \qquad \text{(by (6.9))}.\end{aligned}$$

Now, (6.18) follows by (6.2).

(ii) As above in the derivation of (6.18) we can show that for any $(t,k) \in [0,T) \times \{k_N(c), ..., [(N+1)/2]\}$ with $d_N^{k+1}(t) \leqslant d_N^k(t) - \alpha$

$$\begin{aligned}&\langle X_N(t), |g'_N(\cdot, t)|^2\rangle \\ &\quad \geqslant N^{-3}\, |V'_N(X_N^k(t)-X_N^{k-1}(t)) + V'_N(X_N^k(t)-X_N^{k+1}(t))|^2 \\ &\quad = N^{-3}\, |V'_N(d_N^k(t)) - V'_N(d_N^{k+1}(t))|^2 \qquad (6.23) \\ &\quad \geqslant N^{-3}\alpha^2 N^6\, |V''_1(c)|^2 \qquad \text{(by (6.9))}.\end{aligned}$$

Hence, by (6.7), $|T_N(\alpha, c)| \leqslant C(N^3\alpha^2\, |V''_1(c)|^2)^{-1}$, and (6.20) follows by the choice of $\alpha_N, \underline{c}_N$.

For $k, l \in \{k_N(\underline{c}_N), ..., [(N+1)/2]+1\}$, $k \leqslant l$, $t \in [0, T] \backslash T_N(\alpha_N, \underline{c}_N)$, we have

$$\begin{aligned} X_N^l(t) &= X_N^k(t) + \sum_{n=k+1}^{l} d_N^n(t) \\ &\geqslant X_N^k(t) + \sum_{n=1}^{l-k} (d_N^k(t) - n\alpha_N) \\ &\geqslant X_N^k(t) + (l-k)\, d_N^k(t) - (l-k)^2\, \alpha_N, \end{aligned}$$

whereas by (6.3)

$$X_N^l(t) \leqslant X_N^k(t) + (l-k)\, d_N^k(t).$$

The last two inequalities and $|d_N^k(t) - d_N^l(t)| \leqslant |k-l|\, \alpha_N$ imply (6.21).

In the proof of Lemma (6.17) we use the regularity of X_N, which is described in Lemma (6.1), in a very decisive way. In particular, (6.3) and the monotonicity properties (6.9) of V_1 allow in (6.22), (6.23) a certain reduction of the interaction to nearest neighbours.

C. *Relative Compactness of the Empirical Processes*

For any $N \in \mathbb{N}$ the function $t \to X_N(t)$ is an element of $C([0, T], \mathscr{P}(\mathbb{R}))$, the space of continuous functions $\underline{\mu}: [0, T] \to \mathscr{P}(\mathbb{R})$, equipped with the metric $|||\underline{\mu} - \underline{\tau}|||_* = \sup_{t \leqslant T} \|\underline{\mu}(t) - \underline{\tau}(t)\|_*$.

LEMMA (6.24). *$\{X_N = X_N(t),\ 0 \leqslant t \leqslant T : N \in \mathbb{N}\}$ is relatively compact in $C([0, T], \mathscr{P}(\mathbb{R}))$.*

Proof. For $0 \leqslant s \leqslant t \leqslant T$ and $f \in C_b^1(\mathbb{R})$ we have by (1.8), (6.7)

$$\begin{aligned} \langle X_N(t) - X_N(s), f \rangle &= -\int_s^t \langle X_N(u), g_N'(\cdot, u) f' \rangle\, du \\ &\leqslant \left(\int_0^T \langle X_N(u), |g_N'(\cdot, u)|^2 \rangle\, du \right)^{1/2} \|f'\|_\infty\, (t-s)^{1/2} \\ &\leqslant C\, \|f'\|_\infty\, (t-s)^{1/2}, \qquad \text{uniformly in } N \in \mathbb{N}, \text{ i.e.,} \end{aligned}$$

$$\|X_N(t) - X_N(s)\|_* \leqslant C(t-s)^{1/2}, \qquad 0 \leqslant s \leqslant t \leqslant T, \text{ uniformly in } N \in \mathbb{N}. \tag{6.25}$$

Moreover, we obtain for $\chi(x) = 1 + x^2$

$$\begin{aligned} \langle X_N(t), \chi \rangle &= \langle X_N(0), \chi \rangle - \int_0^t \langle X_N(s), g_N'(\cdot, s)\, \chi' \rangle\, ds \\ &\leqslant \langle X_N(0), \chi \rangle + \int_0^t \langle X_N(s), |g_N'(\cdot, s)|^2 \rangle\, ds \\ &\quad + C \int_0^t \langle X_N(s), \chi \rangle\, ds, \end{aligned}$$

i.e., by Gronwall's inequality and (6.7), (6.15),

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{t\leqslant T} \langle X_N(t), \chi\rangle \\
&\quad \leqslant \sup_{N\in\mathbb{N}} \left(\langle X_N(0), \chi\rangle + \int_0^T \langle X_N(s), |g'_N(\cdot, s)|^2\rangle \, ds \right) \exp(CT) \\
&\quad < \infty.
\end{aligned} \tag{6.26}$$

It is known that any set $\mathscr{P}^a = \{\mu \in \mathscr{P}(\mathbb{R}) : \langle \mu, \chi\rangle \leqslant a\}$, $a > 0$, is compact in $\mathscr{P}(\mathbb{R})$. Therefore, we can apply (6.25), (6.26) and the Arzelà–Ascoli Theorem to finish the proof of Lemma (6.24).

Next, we have to describe the limit process X_∞ of any convergent subsequence $\{X_{N_k} : k \in \mathbb{N}\}$, i.e., to identify its dynamics. For simplicity, we omit the index k in the sequel.

D. *The Dynamics of X_N in a Different Representation*

In this subsection we derive different approximations of (1.8), which describes the measure-valued process X_N exactly. These approximations will be the starting point for the identification of the limit dynamics of X_N. We begin with an exact formulation of (1.10).

LEMMA (6.27).

$$\begin{aligned}
&\lim_{N\to\infty} \sup_{t\leqslant T} \Bigg| \langle X_N(t), f\rangle - \langle X_N(0), f\rangle \\
&\qquad - \frac{1}{2}\int_0^t \frac{1}{N^2} \sum_{\substack{k,l=1\\ k\neq l}}^{N} V_N^{\#}(X_N^k(s) - X_N^l(s))\, f''(X_N^k(s))\, ds \Bigg| = 0, \\
&\qquad f \in C_b^3(\mathbb{R}),
\end{aligned} \tag{6.28}$$

where we used the abbreviations $V_1^{\#}(x) = (-x)\, V_1'(x)$ *and* $V_N^{\#}(x) = N V_1^{\#}(Nx)$ *for* $x \neq 0$.

Proof. As in (1.10) we have

$$\begin{aligned}
&\langle X_N(s), g'_N(\cdot, s) f'\rangle \\
&\quad = \frac{1}{2}\frac{1}{N^2} \sum_{\substack{k,l=1\\ k\neq l}}^{N} V'_N(X_N^k(s) - X_N^l(s))[f'(X_N^k(s)) - f'(X_N^l(s))]
\end{aligned}$$

$$=\frac{1}{2}\frac{1}{N^2}\sum_{\substack{k,l=1\\k\neq l}}^{N} V'_N(X_N^k(s)-X_N^l(s))(X_N^k(s)-X_N^l(s))f''(X_N^k(s))$$

$$-\frac{1}{4}\frac{1}{N^2}\sum_{\substack{k,l=1\\k\neq l}}^{N} V'_N(X_N^k(s)-X_N^l(s))(X_N^k(s)-X_N^l(s))^2$$

$$\times f'''(X_N^k(s)+\theta(N,s,k,l)(X_N^l(s)-X_N^k(s))) \tag{6.29}$$

with $|\theta(...)|\leqslant 1$.
By (6.5), (6.8) the absolute value of the second term on the right side of (6.29) is less than

$$C\,\|f'''\|_\infty\frac{1}{N^2}\sum_{k,l=1}^{N}\frac{1}{1+N\,|X_N^k(s)-X_N^l(s)|}\leqslant\frac{C}{N}\sum_{l=-N}^{N}\frac{1}{1+d_0\,|l|}\leqslant C\frac{\log N}{N}.$$

Now, (6.28) follows from (1.8), (6.29), and the obvious relation $-xV'_N(x)=V_N^{\#}(x)$.

To save some notation we assume from now on without restricting generality in addition to (6.8)

$$\int_{\mathbb{R}} V_1(x)\,dx=1. \tag{6.30}$$

In this case $g_N(\cdot,t)$ is a probability density on $\mathbb{R}$, which by Lemma (5.5) satisfies

$$\lim_{N\to\infty}\sup_{t\leqslant T}|\langle X_N(t),f\rangle-\langle g_N(\cdot,t),f\rangle|=0,\qquad f\in C_b^1(\mathbb{R}). \tag{6.31}$$

For the further investigation of (6.28) we need:

LEMMA (6.32). *Let $b_N=g_N$ or P_N, where $P_N(x,t)=(1/N)\sum_{k=2}^{N}d_N^k(t)^{-1}\mathbb{1}_{(X_N^{k-1}(t),X_N^k(t)]}(x)$. Then*

$$\sup_{N\in\mathbb{N},t\leqslant T}\|b_N(\cdot,t)\|_p<\infty,\qquad p\in[1,\infty]. \tag{6.33}$$

Furthermore, the measure $X_\infty(t)(dx)\,dt$ on $\mathbb{R}\times[0,T]$ has a density x_∞ with respect to Lebesgue measure, and

$$\lim_{N\to\infty}b_N=x_\infty\qquad\text{weakly in } L^2(\mathbb{R}\times[0,T]). \tag{6.34}$$

Proof. The uniform L^∞-boundedness of g_N, resp. P_N, has been stated in (6.14), resp. follows from (6.5). The uniform L^1-boundedness of these functions is implied by their positivity,

$$\int_{\mathbb{R}} g_N(y,t)\,dy=1,$$

and

$$\int_{\mathbb{R}} P_N(y,t)\,dy = \int_{\mathbb{R}} \frac{1}{N} \sum_{k=2}^{N} d_N^k(t)^{-1}\, \mathbb{1}_{(X_N^{k-1}(t),\, X_N^k(t)]}(y)\,dy = \frac{N-1}{N}.$$

The L^p-estimates for $p \in (1, \infty)$ follow by interpolation.

To finish the proof of Lemma (6.32) we have to show by (6.31), (6.33)

$$\lim_{N\to\infty} \langle P_N(\cdot,t), f\rangle = \langle X_\infty(t), f\rangle, \qquad t\in[0,T], \quad f\in C_b^1(\mathbb{R}). \quad (6.35)$$

First,

$$\begin{aligned}\langle P_N(\cdot,t), f\rangle &= \frac{1}{N}\sum_{k=2}^{N}\int_{X_N^{k-1}(t)}^{X_N^k(t)} d_N^k(t)^{-1} f(y)\,dy\\ &= \frac{1}{N}\sum_{k=2}^{N} f(X_N^k(t))\\ &\quad + \frac{1}{N}\sum_{k=2}^{N}\int_{X_N^{k-1}(t)}^{X_N^k(t)} d_N^k(t)^{-1}\,[f(y)-f(X_N^k(t))]\,dy. \quad (6.36)\end{aligned}$$

Next, we obtain

$$\begin{aligned}&\left|\frac{1}{N}\sum_{k=2}^{N}\int_{X_N^{k-1}(t)}^{X_N^k(t)} d_N^k(t)^{-1}\,[f(y)-f(X_N^k(t))]\,dy\right|\\ &\quad\leqslant 2\,\|f\|_\infty \frac{|K_N(\varsigma_N)|}{N} + \frac{1}{N}\|f'\|_\infty \sum_{k\in\{2,\ldots,N\}\setminus K_N(\varsigma_N)} d_N^k(t)\\ &\quad\leqslant C\,\|f\|_\infty (\varsigma_N)^{-2/3} + \|f'\|_\infty\, \varsigma_N/N \qquad \text{(by (6.6))}. \quad (6.37)\end{aligned}$$

Since $|(1/N)\sum_{k=2}^{N} f(X_N^k(t)) - \langle X_N(t), f\rangle| = (1/N)\,|f(X_N^1(t))| \leqslant \|f\|_\infty/N$, (6.35) is a consequence of (6.19), (6.36), (6.37), and Lemma (6.24).

The property of local equilibrium, in particular (6.21), gives a description of the relative positions of neighbouring particles. As indicated in the introduction (cf. (1.11), (1.12)), this can be used to simplify (6.28). However, when applying (6.21) we have to exclude the set of those particles and moments of time $K_N(\varsigma_N)\times[0,T]\cup\{1,\ldots,N\}\times T_N(\alpha_N,\varsigma_N)$, where (6.21) does not apply. Fortunately, (6.6), (6.20) imply that this set is small. Therefore, we can arrive at the following result.

LEMMA (6.38).

$$\begin{aligned}\lim_{N\to\infty}\sup_{t\leqslant T}\Big|&\langle X_N(t), f\rangle - \langle X_N(0), f\rangle\\ &-\frac{1}{2}\int_0^t \langle g_N(\cdot,s),\, F[V_1, P_N(\cdot,s)]f''\rangle\,ds\Big| = 0, \qquad f\in C_b^3(\mathbb{R}). \quad (6.39)\end{aligned}$$

Proof. Let us first remark the following consequences of (1.6), (6.8), (6.9).

$$0 \leqslant F[V_1, p] \leqslant Cp, \qquad p \in (0, \infty), \tag{6.40}$$

$$\|F'[V_1, \cdot]\|_\infty \leqslant C. \tag{6.41}$$

Next, we introduce the additional notation

$$K'_N(\underline{c}_N) = \{k \in \{2, ..., N\} \backslash K_N(\underline{c}_N) \colon \inf\{|k-l| : l \in K_N(\underline{c}_N)\} > N^{1/2}\},$$

$$\underline{X}_N(t) = \frac{1}{N} \sum_{k \in K'_N(\underline{c}_N)} \delta_{X^k_N(t)},$$

and

$$\mathbf{R}_{N,k} = \{l \in \{1, ..., N\} : 0 < |k-l| \leqslant N^\varepsilon\}.$$

Relations (6.5), (6.8) imply

$$\sup_{N \in \mathbb{N}, t \geqslant 0, x \in \mathbb{R}} \left| \frac{1}{N} \sum_{l=1}^{N} V^{\#}_N(x - X^l_N(t)) \right| \leqslant C \sum_{n=-\infty}^{\infty} (1 + |n|\, d_0)^{-2} < \infty, \tag{6.42}$$

and similarly

$$\limsup_{N \to \infty} \sup_{t \geqslant 0, x \in \mathbb{R}} \left| \frac{1}{N} \sum_{\{l : |X^l_N(t) - x| > N^{\varepsilon/2 - 1}\}} V^{\#}_N(x - X^l_N(t)) \right| = 0. \tag{6.43}$$

From (6.5), (6.6), (6.20), (6.28), (6.42), (6.43) we conclude

$$\lim_{N \to \infty} \sup_{t \leqslant T} \Bigg| \langle X_N(t), f \rangle - \langle X_N(0), f \rangle$$
$$- \frac{1}{2} \int_{[0,t] \backslash T_N(\alpha_N, \underline{c}_N)} \frac{1}{N^2} \sum_{k \in K'_N(\underline{c}_N)} \sum_{l \in \mathbf{R}_{N,k}} V^{\#}_N(X^k_N(s) - X^l_N(s)) f''(X^k_N(s))\, ds \Bigg| = 0,$$

$$f \in C^3_b(\mathbb{R}). \tag{6.44}$$

Next, we have for fixed $s \in [0, T] \backslash T_N(\alpha_N, \underline{c}_N)$

$$\Bigg| \frac{1}{N^2} \sum_{k \in K'_N(\underline{c}_N)} \sum_{l \in \mathbf{R}_{N,k}} V^{\#}_N(X^k_N(s) - X^l_N(s)) f''(X^k_N(s))$$
$$- \langle \underline{X}_N(s), F[V_1, P_N(\cdot, s)] f'' \rangle \Bigg|$$

$$= \left| \frac{1}{N} \sum_{k \in K'_N(\underline{c}_N)} f''(X_N^k(s)) \left(\frac{1}{N} \sum_{l \in \mathbf{R}_{N,k}} V_N^{\#}(X_N^k(s) - X_N^l(s)) \right.\right.$$

$$\left.\left. - \frac{1}{N} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} V_N^{\#}(l d_N^k(s)) \right) \right|$$

$$\leqslant \|f''\|_\infty \frac{C}{N^2} \sum_{k \in K'_N(\underline{c}_N)} \sum_{l \in \mathbf{R}_{N,k}} |V_N^{\#}(X_N^k(s) - X_N^l(s)) - V_N^{\#}((l-k)\, d_N^k(s))|$$

$$+ \|f''\|_\infty \frac{C}{N^2} \sum_{k \in K'_N(\underline{c}_N)} \sum_{|l| > N^\varepsilon} |V_N^{\#}(l d_N^k(s))|.$$

By (6.5), (6.8) the second term tends to 0 as $N \to \infty$ uniformly in s. Then we note that any k or l occurring in the first term belongs to $\{2, ..., N\} \backslash K_N(\underline{c}_N)$, and therefore we can conclude from (6.8), (6.21) that this expression is less than $CN^{-2}N^{1+\varepsilon} \|(V_N^{\#})'\|_\infty N^{2\varepsilon}N^{\varepsilon - 3/2} \leqslant CN^{4\varepsilon - 1/2}$ uniformly in s. Hence,

$$\lim_{N \to \infty} \int_{[0,T] \backslash T_N(\alpha_N, \underline{c}_N)} \left| \frac{1}{N^2} \sum_{k \in K'_N(\underline{c}_N)} \sum_{l \in \mathbf{R}_{N,k}} V_N^{\#}(X_N^k(s) - X_N^l(s)) f''(X_N^k(s)) \right.$$

$$\left. - \langle \underline{X}_N(s), F[V_1, P_N(\cdot, s)] f'' \rangle \right| ds = 0, \qquad f \in C_b^2(\mathbb{R}). \tag{6.45}$$

Now, we obtain for $f \in C_b^3(\mathbb{R})$ uniformly in $s \in [0, T] \backslash T_N(\alpha_N, \underline{c}_N)$

$$|\langle \underline{X}_N(s), F[V_1, P_N(\cdot, s)] f'' \rangle - \langle g_N(\cdot, s), F[V_1, P_N(\cdot, s)] f'' \rangle|$$

$$\leqslant \int_{\mathbb{R}} \underline{X}_N(s)(dx) \int_{\mathbb{R}} dy\, V_N(x - y)$$

$$\times |F[V_1, P_N(x, s)] f''(x) - F[V_1, P_N(y, s)] f''(y)|$$

$$+ \frac{1}{N} \sum_{k \in \{1, ..., N\} \backslash K'_N(\underline{c}_N)} \int_{\mathbb{R}} dy\, V_N(X_N^k(s) - y)$$

$$\times F[V_1, P_N(y, s)]\, |f''(y)| \qquad \text{(by (6.30))}$$

$$\leqslant \frac{1}{N} \sum_{k \in K'_N(\underline{c}_N)} \sum_{l = k - [N^\varepsilon]}^{k + [N^\varepsilon]} \int_{\mathbb{R}} dy\, V_N(X_N^k(s) - y)\, \mathbb{1}_{(X_N^{l-1}(s), X_N^l(s)]}(y)$$

$$\times \left| F\left[V_1, \frac{1}{N d_N^k(s)} \right] f''(X_N^k(s)) - F\left[V_1, \frac{1}{N d_N^l(s)} \right] f''(y) \right|$$

$$+ C \|f''\|_\infty \int_{\{|z| \geqslant d_0 [N^\varepsilon]/N\}} V_N(z)\, dz + \frac{C}{N} (|K_N(\underline{c}_N)| + N^{1/2}) \|f''\|_\infty$$

$$\text{(by (6.5), (6.8), (6.33), (6.40)).} \tag{6.46}$$

By (6.6), (6.8) the second and third term on the right side of (6.46) vanish in the limit $N \to \infty$, whereas the first term is less than

$$\frac{C}{N} \sum_{k \in K'_N(\underline{c}_N)} N^{\varepsilon} \sup_{l = k - [N^{\varepsilon}], \dots, k + [N^{\varepsilon}]} \frac{1}{N} \left| \frac{1}{d_N^k(s)} - \frac{1}{d_N^l(s)} \right| \|f''\|_{\infty}$$

$$+ C \int_{\mathbb{R}} \underline{X}_N(s)(dx) \int_{\mathbb{R}} dy\, V_N(x-y)\, |f''(x) - f''(y)|$$

$$\text{(by (6.33), (6.40), (6.41))}$$

$$\leqslant \frac{C}{N} \sum_{k \in K'_N(\underline{c}_N)} N^{\varepsilon} \sup_{l = k - [N^{\varepsilon}], \dots, k + [N^{\varepsilon}]} N\, |d_N^k(s) - d_N^l(s)|$$

$$+ C \int_{\mathbb{R}} dz\, V_N(z) \min\{\|f''\|_{\infty}, \|f'''\|_{\infty} |z|\} \qquad \text{(by (6.5))}$$

$$\leqslant C \left(N^{1+\varepsilon} N^{\varepsilon} \alpha_N + \int_{\mathbb{R}} dz\, V_1(z) \min\{1, |z|/N\} \right),$$

since any k or l occurring in the sum above belongs to $\{1, \dots, N\} \backslash K_N(\underline{c}_N)$. Since this expression tends to 0 as $N \to \infty$, and since by (6.20), (6.30), (6.33), (6.40)

$$\lim_{N \to \infty} \left| \int_{T_N(\alpha_N, \underline{c}_N)} \langle g_N(\cdot, s), F[V_1, P_N(\cdot, s)] f'' \rangle\, ds \right|$$

$$\leqslant C \|f''\|_{\infty} \lim_{N \to \infty} \|P_N\|_{\infty} |T_N(\alpha_N, \underline{c}_N)| = 0,$$

the proof of Lemma (6.38) is complete.

E. *A L^2-Convergence Result*

In (6.39) the empirical process X_N enters in a nonlinear way. Since the integral term does not depend continuously on $X_N \in C([0, T], \mathscr{P}(\mathbb{R}))$, we need for the study of its asymptotic behaviour some stronger form of relative compactness than that of Lemma (6.24). In particular, we shall utilize the property of local equilibrium (Lemma (6.17)) to show that P_N converges strongly in $L^2(\mathbb{R} \times [0, T])$.

LEMMA (6.47).

$$\lim_{N \to \infty} \|P_N - x_{\infty}\|_2 = 0. \tag{6.48}$$

In (6.48) and its proof $\|\cdot\|_2$ denotes the norm in $L^2(\mathbb{R} \times [0, T])$.

Proof. Let us define for $\delta \in (0, 1)$

$$P_{N,\delta}(x, t) = \sum_{k=k_N(1/\delta, t)}^{N-k_N(1/\delta, t)+2} \mathbb{1}_{(X_N^{k-1}(t), X_N^k(t)]}(x) \frac{1}{Nd_N^k(t)}.$$

Then we obtain for $N, N' \in \mathbb{N}$, $N \leqslant N'$

$$\|P_N - P_{N'}\|_2 \leqslant 2 \sup_{M \in \mathbb{N}} \|P_M - P_{M,\delta}\|_2 + \|P_{N,\delta} - P_{N',\delta}\|_2. \tag{6.49}$$

Moreover, we have for fixed $A \geqslant 1$

$$\begin{aligned}\|P_{N,\delta} - P_{N',\delta}\|_2^2 &\leqslant \int_0^T dt \int_{\{|\mu| \leqslant A\}} d\mu\, |\widetilde{P_{N,\delta}}(\mu, t) - \widetilde{P_{N',\delta}}(\mu, t)|^2 \\ &\quad + 4 \sup_{M \in \mathbb{N}} \int_0^T dt \int_{\{|\mu| > A\}} d\mu\, |\widetilde{P_{M,\delta}}(\mu, t)|^2 \\ &\leqslant C \Bigg(\int_0^T dt \int_{\{|\mu| \leqslant A\}} d\mu\, |\widetilde{P_N}(\mu, t) - \widetilde{P_{N'}}(\mu, t)|^2 \\ &\quad + \sup_{M \in \mathbb{N}} \Bigg\{ \|P_{M,\delta} - P_M\|_2^2 \\ &\quad + \frac{1}{A^\varepsilon} \int_0^T dt \int_{\{|\mu| \geqslant 1\}} d\mu\, |\mu|^\varepsilon\, |\widetilde{P_{M,\delta}}(\mu, t)|^2 \Bigg\} \Bigg).\end{aligned} \tag{6.50}$$

Let us investigate the asymptotic behaviour as $N, N' \to \infty$ of the different terms in (6.49), (6.50).

We obtain by (6.2), (6.3), (6.6)

$$\begin{aligned}\|P_M - P_{M,\delta}\|_2^2 &\leqslant \int_0^T dt \sum_{k \in K_M(1/\delta, t)} \int_{X_M^{k-1}(t)}^{X_M^k(t)} dx \left| \frac{1}{Md_M^k(t)} \right|^2 \\ &\qquad (K_M(c, t) = \{k \in \{2, ..., M\} : Md_M^k(t) \geqslant c\}) \\ &\leqslant \frac{C}{M} \int_0^T dt \sum_{k \in K_M(1/\delta, t)} \frac{1}{Md_M^k(t)} \leqslant \frac{C}{M} |K_M(1/\delta)|\, \delta \\ &\leqslant C\delta^{5/3}, \qquad \text{uniformly in } M \in \mathbb{N}.\end{aligned} \tag{6.51}$$

Next, we have for $\mu \neq 0$ uniformly in M

$$\begin{aligned}&|\widetilde{P_{M,\delta}}(\mu, t)| \\ &\quad = (2\pi)^{-1/2} \left| \sum_{k=k_M(1/\delta, t)}^{M-k_M(1/\delta, t)+2} \frac{1}{Md_M^k(t)} \int_{X_M^{k-1}(t)}^{X_M^k(t)} \exp(i\mu x)\, dx \right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{1/2}\,|\mu|} \\
&\quad \times \left| \sum_{k=k_M(1/\delta,t)}^{M-k_M(1/\delta,t)+2} \frac{1}{M d_M^k(t)} \left(\exp(i\mu X_M^k(t)) - \exp(i\mu X_M^{k-1}(t))\right) \right| \\
&= \frac{1}{(2\pi)^{1/2}\,|\mu|} \\
&\quad \times \left| \sum_{k=k_M(1/\delta,t)}^{M-k_M(1/\delta,t)+1} \exp(i\mu X_M^k(t)) \frac{1}{M} \left(\frac{1}{d_M^k(t)} - \frac{1}{d_M^{k+1}(t)} \right) \right. \\
&\quad - \exp(i\mu X_M^{k_M(1/\delta,t)-1}(t)) \frac{1}{M d_M^{k_M(1/\delta,t)}(t)} \\
&\quad \left. + \exp(i\mu X_M^{M-k_M(1/\delta,t)+2}(t)) \frac{1}{M d_M^{M-k_M(1/\delta,t)+2}(t)} \right| \\
&\leqslant \frac{C}{|\mu|} \left(\frac{1}{M} \sum_{k=k_M(1/\delta,t)}^{M-k_M(1/\delta,t)+1} M^4\, |d_M^k(t) - d_M^{k+1}(t)|^2 + 1 \right)^{1/2} \\
&\qquad \text{(by (6.5))}.
\end{aligned}
$$

This estimate and (6.18) imply

$$
\begin{aligned}
&\int_0^T dt \int_{\{|\mu| \geqslant 1\}} d\mu\, |\mu|^{\varepsilon}\, |\widetilde{P_{M,\delta}}(\mu, t)|^2 \\
&\qquad \leqslant C \int_{\{|\mu| \geqslant 1\}} d\mu\, |\mu|^{\varepsilon - 2} \int_0^T dt \\
&\qquad\quad \times \left(\frac{1}{M} \sum_{k=k_M(1/\delta,t)}^{M-k_M(1/\delta,t)+1} M^4\, |d_M^k(t) - d_M^{k+1}(t)|^2 + 1 \right) \\
&\qquad \leqslant C(V_1''(1/\delta)^{-2} + 1), \qquad \text{uniformly in } M \in \mathbb{N}. \qquad (6.52)
\end{aligned}
$$

Finally, we obtain by (6.34) for any $A \geqslant 1$

$$
\begin{aligned}
&\lim_{N,N' \to \infty} \int_0^T dt \int_{\{|\mu| \leqslant A\}} d\mu\, |\widetilde{P_N}(\mu, t) - \widetilde{P_{N'}}(\mu, t)|^2 \\
&\qquad = C \lim_{N,N' \to \infty} \int_0^T dt \int_{\{|\mu| \leqslant A\}} d\mu \\
&\qquad\quad \times |\langle P_N(\cdot, t), \exp(i\mu \cdot)\rangle - \langle P_{N'}(\cdot, t), \exp(i\mu \cdot)\rangle|^2 = 0. \qquad (6.53)
\end{aligned}
$$

Inequalities (6.49)–(6.52) imply

$$\|P_N - P_{N'}\|_2^2 \leqslant C\left(\delta^{5/3} + \frac{1}{A^{\varepsilon}}(V_1''(1/\delta)^{-2}+1) + \int_0^T dt \int_{\{|\mu| \leqslant A\}} d\mu \, |\widetilde{P_N}(\mu, t) - \widetilde{P_{N'}}(\mu, t)|^2\right).$$

By (6.53) this expression can be made arbitrarily small, if we first choose δ, resp. A, sufficiently small, resp. large, and then let N, N' tend to ∞. This fact and (6.34) complete the proof of Lemma (6.47).

F. *Identification of the Limit Dynamics and Completion of the Proof of Theorem* (3.8)

The results obtained so far allow us to finish the proof of Theorem (3.8) fairly quickly.

First, we obtain for $f \in C_b^3(\mathbb{R})$

$$\begin{aligned}
&\lim_{N\to\infty} \sup_{t \leqslant T} \left| \int_0^t \langle g_N(\cdot, s), F[V_1, P_N(\cdot, s)] f'' \rangle \, ds - \int_0^t \langle x_\infty(\cdot, s), F[V_1, x_\infty(\cdot, s)] f'' \rangle \, ds \right| \\
&\leqslant \lim_{N\to\infty} \sup_{t \leqslant T} \left| \int_0^t \langle g_N(\cdot, s), (F[V_1, P_N(\cdot, s)] - F[V_1, x_\infty(\cdot, s)]) f'' \rangle \, ds + \int_0^t \langle g_N(\cdot, s) - x_\infty(\cdot, s), F[V_1, x_\infty(\cdot, s)] f'' \rangle \, ds \right| \\
&\leqslant \lim_{N\to\infty} \left\{ \|g_N\|_2 \, \|F'[V_1, \cdot]\|_\infty \, \|P_N - x_\infty\|_2 \, \|f''\|_\infty + \left| \int_0^t \langle g_N(\cdot, s) - x_\infty(\cdot, s), F[V_1, x_\infty(\cdot, s)] f'' \rangle \, ds \right| \right\} = 0
\end{aligned}$$

(by (6.33), (6.34), (6.40), (6.41), (6.48)). (6.54)

Hence, we obtain by (3.7), (6.39), and Lemma (6.24)

$$\langle x_\infty(\cdot, t), f \rangle = \langle p_0, f \rangle + \frac{1}{2} \int_0^t \langle x_\infty(\cdot, s), F[V_1, x_\infty(\cdot, s)] f'' \rangle \, ds, \quad f \in C_b^2(\mathbb{R}),\ t \in [0, T]. \tag{6.55}$$

Note that we have used here the fact that the validity of (6.55) for $f \in C_b^3(\mathbb{R})$, which is implied by (3.7), (6.39), and Lemma (6.24), yields its validity for any $f \in C_b^2(\mathbb{R})$.

By (6.33), (6.55), x_∞ is a solution of (1.14) in $L^1(\mathbb{R} \times [0, T]) \cap L^\infty(\mathbb{R} \times [0, T])$. Hence, to finish the proof of Theorem (3.8) we only need:

LEMMA (6.56). *Equation* (1.14) *has a unique solution in* $L^1(\mathbb{R} \times [0, T]) \cap L^\infty(\mathbb{R} \times [0, T])$.

Proof. By (1.6), (3.2), (6.9) the function $\underline{\beta}(p) = pF[V_1, p]$, $p \geqslant 0$, is differentiable and satisfies $\underline{\beta}(0) = 0$, resp. $\underline{\beta}'(p) > 0$. Moreover, (6.33), (6.34), (6.40) imply $(x, t) \to \underline{\beta}(x_\infty(x, t)) \in L^1(\mathbb{R} \times [0, T])$. Hence, Lemma (6.56) follows from [18].

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